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Fuzzy-Initialized Logistic Dynamics for Macroeconomic Business Cycles: A Topological Framework

Abstract

We develop a mathematical framework connecting fuzzy logic inference with deterministic nonlinear dynamics for macroeconomic modeling. Starting from linguistic policy variables encoded via continuous fuzzy membership functions, we prove that the state space inherits a canonical compact metrizable topology. Using Takagi-Sugeno fuzzy inference, we construct a well-posed ordinary differential equation and establish topological conjugacy with multidimensional logistic dynamics. We characterize the stability properties of fixed points and prove existence, uniqueness, and continuous dependence of solutions. The framework enables analytical treatment of qualitative economic communications within dynamical systems theory, with applications to inflation forecasting and policy design.

Index Terms— *fuzzy logic, business cycles, nonlinear dynamics, Takagi-Sugeno systems, topological conjugacy*

I INTRODUCTION

Macroeconomic phenomena are driven by qualitative expectations communicated through linguistic statements. Central bank announcements employ imprecise terms such as "moderate growth" or "accommodative stance," and market participants form beliefs based on such communications. Fuzzy logic, introduced by Zadeh [1], provides a mathematical framework for reasoning with such linguistic variables, while nonlinear dynamics [2] generates endogenous oscillations that resemble observed business cycles.

This paper establishes a rigorous connection between these two frameworks. We prove that continuous fuzzy membership functions induce a metrizable topology on the state space, that Takagi-Sugeno inference produces a well-posed differential equation, and that topological conjugacy with logistic dynamics enables complete stability analysis. The resulting theory provides analytical tools for understanding how linguistic economic communications translate into deterministic dynamical behavior.

The paper proceeds as follows. Section II develops the topological foundations of fuzzy state spaces. Section III constructs the Takagi-Sugeno dynamical system and proves well-posedness. Section IV establishes conjugacy with logistic dynamics and analyzes stability. Section V discusses economic interpretations and applications.

II TOPOLOGICAL STRUCTURE OF FUZZY STATE SPACES

A Membership Functions and the Embedding Map

We begin by establishing the topological structure induced by fuzzy membership functions on the state space.

Definition 1. *Let $U = [0,1]^n$ denote the fuzzy state space equipped with the standard Euclidean topology. A membership system is a finite collection $\mathcal{F} = \{\mu_1, \ldots, \mu_p\}$ of continuous functions $\mu_k : U \to [0,1]$. Each μ_k represents a linguistic variable such as "high inflation" or "tight credit conditions."*

Definition 2. *The membership embedding associated with $\mathcal{F}$ is the map*

$$\Phi : U \to [0,1]^p, \quad \Phi(x) = (\mu_1(x), \ldots, \mu_p(x)).$$

The image $M = \Phi(U) \subset [0,1]^p$ is the membership space.

The embedding Φ encodes the state x through its membership degrees in the various fuzzy sets. We now establish that this encoding preserves the topological structure when Φ is injective.

Theorem 1 (Metrization of Fuzzy State Spaces)**.** *Suppose $\Phi : U \to [0,1]^p$ is injective. Then $\Phi : U \to M$ is a homeomorphism, and the function*

$$d_\Phi(x, y) = \|\Phi(x) - \Phi(y)\|_2$$

defines a metric on U that generates the standard topology. In particular, (U, d_Φ) is a compact metric space.

Proof. We verify the hypotheses of the theorem on continuous bijections from compact spaces to Hausdorff spaces.

Since each μ_k is continuous and the product topology on $[0,1]^p$ is generated by projections, the map Φ is continuous. By assumption, Φ is injective. The space $U = [0,1]^n$ is compact as a closed bounded subset of $\mathbb{R}^n$, and $[0,1]^p$ is Hausdorff. A continuous injection from a compact space to a Hausdorff space is a homeomorphism onto its image. Therefore $\Phi : U \to M$ is a homeomorphism.

The image $M = \Phi(U)$ is compact as the continuous image of a compact set. The function $d_\Phi(x, y) = \|\Phi(x) - \Phi(y)\|_2$ is the pullback of the Euclidean metric on $[0,1]^p$ via Φ. Since Φ is a homeomorphism, this pullback metric

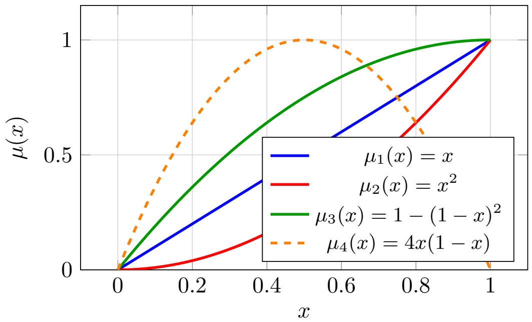

Figure 1: Four membership functions on $[0, 1]$. The functions μ_1, μ_2, μ_3 are strictly monotone and encode linguistic variables such as "low," "very low," and "high." The function μ_4 is unimodal and encodes "medium."

generates the same topology as the original topology on U. Compactness of (U, d_Φ) follows from compactness of U in its original topology. $\square$

The injectivity assumption on Φ is essential and has a natural interpretation: the membership system $\mathcal{F}$ must be rich enough to distinguish all states. We provide a sufficient condition.

Proposition 2. *Suppose there exist indices $k_1, \ldots, k_n \in \{1, \ldots, p\}$ and continuous strictly monotone functions $g_i : [0, 1] \to [0, 1]$ such that $\mu_{k_i}(x) = g_i(x_i)$ for each $i = 1, \ldots, n$. Then Φ is injective.*

Proof. Let $x, y \in U$ with $\Phi(x) = \Phi(y)$. Then $\mu_{k_i}(x) = \mu_{k_i}(y)$ for each i, which gives $g_i(x_i) = g_i(y_i)$. Since each g_i is strictly monotone, it is injective, so $x_i = y_i$ for all i. Thus $x = y$. $\square$

Example 1. *Consider the membership system consisting of $\mu_i(x) = x_i$ for $i = 1, \ldots, n$ together with interaction terms $\mu_{ij}(x) = x_i x_j$ for $i < j$. The identity functions $g_i(t) = t$ are strictly monotone, so by Proposition 2, the embedding is injective. The metric d_Φ then incorporates both individual variable values and their pairwise products, capturing interaction effects in the distance function.*

B The Mamdani Inference Map

We now analyze the Mamdani inference procedure, which aggregates fuzzy rules to produce a crisp output via defuzzification.

Definition 3. *A Mamdani fuzzy system consists of r rules of the form*

$$\text{Rule } j: \ \text{IF } x \text{ is } A_j \text{ THEN } y \text{ is } B_j$$

where A_j and B_j are fuzzy sets with membership functions $\alpha_j : U \to [0, 1]$ and $\beta_j : [0, 1] \to [0, 1]$ respectively. The aggregated output membership function is

$$G(x, y) = \max_{j=1}^{r} \min(\alpha_j(x), \beta_j(y)).$$

The Mamdani inference map is the centroid defuzzification

$$\mathcal{I}(x) = \frac{\int_0^1 y\, G(x, y)\, dy}{\int_0^1 G(x, y)\, dy}.$$

Assumption 1. *There exist constants $0 < m \le M < \infty$ such that for all $x \in U$,*

$$m \le \int_0^1 G(x, y)\, dy \le M \quad \text{and} \quad \sup_{y \in [0,1]} G(x, y) \le M.$$

The lower bound $m > 0$ ensures that the denominator in the centroid formula is bounded away from zero, preventing singularities.

Theorem 3 (Continuity of Mamdani Inference)**.** *Under Assumption 1, the Mamdani inference map $\mathcal{I} : U \to [0, 1]$ is continuous.*

Proof. We establish continuity by showing that $\mathcal{I}(x_k) \to \mathcal{I}(x)$ whenever $x_k \to x$ in U.

For each rule j, the membership function α_j is continuous, so $\alpha_j(x_k) \to \alpha_j(x)$ as $k \to \infty$. Since the minimum function is continuous on $\mathbb{R}^2$ and the maximum of finitely many continuous functions is continuous, we have

$$G(x_k, y) \to G(x, y) \quad \text{for each } y \in [0, 1].$$

Moreover, $|G(x_k, y)| \le M$ for all k and y by Assumption 1. By the dominated convergence theorem,

$$\int_0^1 G(x_k, y)\, dy \to \int_0^1 G(x, y)\, dy$$

and

$$\int_0^1 y\, G(x_k, y)\, dy \to \int_0^1 y\, G(x, y)\, dy.$$

Define the numerator $N(x) = \int_0^1 y\, G(x, y)\, dy$ and denominator $D(x) = \int_0^1 G(x, y)\, dy$. We have shown $N(x_k) \to N(x)$ and $D(x_k) \to D(x)$. Since $D(x) \ge m > 0$ for all x, the quotient $\mathcal{I}(x) = N(x)/D(x)$ is continuous. $\square$

Corollary 4. *The image $\mathcal{I}(U)$ is a compact subset of $[0, 1]$.*

III TAKAGI-SUGENO DYNAMICAL SYSTEMS

A Construction of the Vector Field

The Takagi-Sugeno (T-S) framework [3] provides a systematic method for constructing nonlinear dynamical systems from fuzzy rules with linear consequents.

Definition 4. *A Takagi-Sugeno fuzzy system of order r on $U = [0,1]^n$ consists of rules*

$$\text{Rule } i: \text{ IF } z(x) \text{ is } M_i \text{ THEN } \dot{x} = A_i x + b_i$$

for $i = 1, \dots, r$, where $z : U \to \mathbb{R}^q$ is a continuous function of premise variables, M_i is a fuzzy set on $\mathbb{R}^q$ with continuous membership function $\nu_i : \mathbb{R}^q \to [0,1]$, and $A_i \in \mathbb{R}^{n\times n}$, $b_i \in \mathbb{R}^n$ are the local linear model parameters.

Definition 5. *The rule activation for rule i at state x is $h_i(x) = \nu_i(z(x))$. The normalized activation weights are*

$$w_i(x) = \frac{h_i(x)}{\sum_{j=1}^r h_j(x)}$$

when $\sum_{j=1}^r h_j(x) > 0$. The T-S vector field is

$$F(x) = \sum_{i=1}^r w_i(x)(A_i x + b_i). \tag{1}$$

Assumption 2. *There exists $\delta > 0$ such that $\sum_{i=1}^r h_i(x) \geq \delta$ for all $x \in U$.*

This nondegeneracy assumption ensures that at least one rule has positive activation at every state, preventing division by zero in the weight normalization.

Theorem 5 (Lipschitz Continuity of T-S Vector Field)**.** *Suppose the premise function z is Lipschitz with constant L_z, each membership function ν_i is Lipschitz with constant L_ν, and Assumption 2 holds. Then the T-S vector field F is Lipschitz continuous on U.*

Proof. We construct the Lipschitz constant through a chain of estimates. Let $x, y \in U$ be arbitrary.

The premise function satisfies $\|z(x) - z(y)\| \leq L_z\|x - y\|$. Each activation function satisfies

$$\begin{aligned} |h_i(x) - h_i(y)| &= |\nu_i(z(x)) - \nu_i(z(y))| \\ &\leq L_\nu \|z(x) - z(y)\| \\ &\leq L_\nu L_z \|x - y\|. \end{aligned}$$

Define $L_h = L_\nu L_z$.

The sum $S(x) = \sum_{j=1}^r h_j(x)$ satisfies

$$|S(x) - S(y)| \leq \sum_{j=1}^r |h_j(x) - h_j(y)| \leq rL_h\|x - y\|.$$

For the normalized weights, we compute

$$\begin{aligned} |w_i(x) - w_i(y)| &= \left|\frac{h_i(x)}{S(x)} - \frac{h_i(y)}{S(y)}\right| \\ &= \left|\frac{h_i(x)S(y) - h_i(y)S(x)}{S(x)S(y)}\right|. \end{aligned}$$

This is bounded by

$$\frac{|h_i(x) - h_i(y)||S(y)| + |h_i(y)||S(x) - S(y)|}{S(x)S(y)}.$$

Using $|h_i| \leq 1$, $|S| \leq r$, and $S(x), S(y) \geq \delta$, we obtain

$$|w_i(x) - w_i(y)| \leq \frac{rL_h + rL_h}{\delta^2}\|x - y\| = \frac{2rL_h}{\delta^2}\|x - y\|.$$

Define $L_w = 2rL_h/\delta^2$.

Let $K = \max_i \|A_i\|$ and $B = \max_{i,x\in U} \|A_i x + b_i\|$, which is finite by compactness of U. Then

$$\begin{aligned} \|F(x) - F(y)\| &\leq \sum_{i=1}^r |w_i(x) - w_i(y)| \cdot \|A_i x + b_i\| \\ &\quad + \sum_{i=1}^r w_i(y)\|A_i(x - y)\| \\ &\leq rL_w B\|x - y\| + K\|x - y\|. \end{aligned}$$

Therefore F is Lipschitz with constant $L_F = rL_w B + K$. ☐

B Well-Posedness and Invariance

We now establish that the T-S system generates a well-defined dynamical system on the state space U.

Assumption 3. *The vector field F satisfies the inward-pointing boundary condition: for each $i = 1, \dots, n$,*

$$\begin{aligned} x_i = 0 &\implies F_i(x) \geq 0, \\ x_i = 1 &\implies F_i(x) \leq 0. \end{aligned}$$

This condition ensures that trajectories cannot escape the unit cube U.

Theorem 6 (Well-Posedness of T-S Systems)**.** *Suppose F satisfies the conditions of Theorem 5 and Assumption 3. Then for each $x_0 \in U$, the initial value problem*

$$\frac{dx}{dt} = F(x), \quad x(0) = x_0$$

has a unique solution $x : [0, \infty) \to U$. The flow map $\phi_t : U \to U$ defined by $\phi_t(x_0) = x(t; x_0)$ is continuous in both t and x_0, and the family $\{\phi_t\}_{t\geq 0}$ forms a continuous semiflow on U.

Proof. Since F is Lipschitz on the compact set U, the Picard-Lindelöf theorem guarantees existence and uniqueness of a maximal solution $x : [0, T_{\max}) \to \mathbb{R}^n$.

We prove that $x(t) \in U$ for all $t \in [0, T_{\max})$ by showing that the boundary of U is not crossed. Suppose for contradiction that $t^* = \inf\{t > 0 : x(t) \notin U\}$ exists and is finite. By continuity, $x(t^*) \in \partial U$, so there exists an index i with either $x_i(t^*) = 0$ or $x_i(t^*) = 1$.

Phase portrait of T-S system

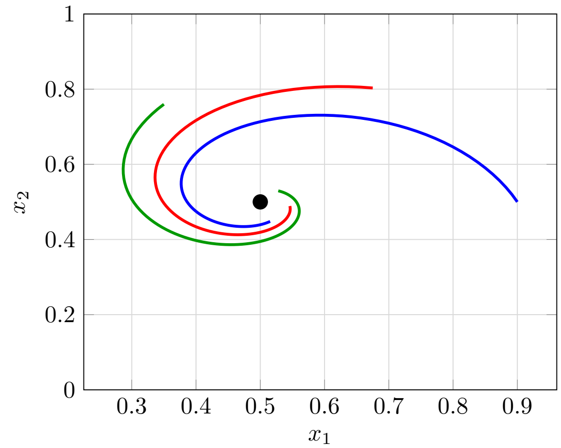

Figure 2: Phase portrait of a two-dimensional T-S system showing trajectories spiraling toward the interior fixed point at $(0.5, 0.5)$. The oscillatory convergence is characteristic of business cycle dynamics.

Consider the case $x_i(t^*) = 0$. By Assumption 3, $F_i(x(t^*)) \geq 0$. For $\epsilon > 0$ sufficiently small, the fundamental theorem of calculus gives

$$x_i(t^* + \epsilon) = x_i(t^*) + \int_{t^*}^{t^*+\epsilon} F_i(x(s))\, ds.$$

By continuity of $F_i \circ x$ and the fact that $F_i(x(t^*)) \geq 0$, for small ϵ the integrand remains nonnegative on $[t^*, t^* + \epsilon]$. Therefore $x_i(t^* + \epsilon) \geq x_i(t^*) = 0$, contradicting the assumption that $x(t)$ exits U through the face $x_i = 0$.

The case $x_i(t^*) = 1$ is analogous: Assumption 3 gives $F_i(x(t^*)) \leq 0$, so $x_i(t^* + \epsilon) \leq 1$.

Since $x(t) \in U$ for all $t < T_{\max}$ and U is compact, the solution cannot blow up in finite time. Therefore $T_{\max} = \infty$.

Continuous dependence on initial conditions follows from Gronwall's inequality. If $x(t) = x(t; x_0)$ and $y(t) = x(t; y_0)$ are solutions with different initial conditions, then

$$\|x(t) - y(t)\| \leq \|x_0 - y_0\| + \int_0^t L_F \|x(s) - y(s)\|\, ds.$$

Gronwall's lemma yields $\|x(t) - y(t)\| \leq \|x_0 - y_0\| e^{L_F t}$, proving continuity of ϕ_t in x_0.

Continuity in t follows from $x(t) = x_0 + \int_0^t F(x(s))\, ds$ and boundedness of F on U.

The semiflow properties $\phi_0 = \mathrm{id}$ and $\phi_{t+s} = \phi_t \circ \phi_s$ follow from uniqueness of solutions. $\square$

IV LOGISTIC DYNAMICS AND CONJUGACY

A The Logistic Vector Field

The logistic equation is a fundamental model of population dynamics that exhibits rich nonlinear behavior. We consider its n-dimensional generalization.

Definition 6. *The coordinatewise logistic vector field on* $U = [0, 1]^n$ *is*

$$L(x) = (r_1 x_1(1 - x_1), \ldots, r_n x_n(1 - x_n))$$

where $r_i > 0$ *are growth parameters. The associated differential equation is*

$$\frac{dx_i}{dt} = r_i x_i(1 - x_i), \quad i = 1, \ldots, n. \tag{2}$$

This system decouples into n independent one-dimensional logistic equations. Each component has the explicit solution

$$x_i(t) = \frac{x_i(0) e^{r_i t}}{1 - x_i(0) + x_i(0) e^{r_i t}}$$

for $x_i(0) \in (0, 1)$.

Proposition 7. *The logistic system* (2) *has* 2^n *fixed points, namely all vertices of the unit cube* $\{0, 1\}^n$*. The origin is unstable, the point* $(1, \ldots, 1)$ *is asymptotically stable, and all other vertices are saddle points.*

Proof. A point x^* is a fixed point if and only if $r_i x_i^*(1 - x_i^*) = 0$ for all i, which requires $x_i^* \in \{0, 1\}$ for each i. This gives 2^n fixed points.

The Jacobian of L at x is the diagonal matrix $J(x) = \mathrm{diag}(r_1(1 - 2x_1), \ldots, r_n(1 - 2x_n))$. At a vertex $x^* \in \{0, 1\}^n$, the eigenvalues are $\lambda_i = r_i(1 - 2x_i^*)$. If $x_i^* = 0$, then $\lambda_i = r_i > 0$ (unstable direction). If $x_i^* = 1$, then $\lambda_i = -r_i < 0$ (stable direction).

At the origin, all eigenvalues are positive, so it is unstable. At $(1, \ldots, 1)$, all eigenvalues are negative, so it is asymptotically stable. At any other vertex, there is at least one positive and one negative eigenvalue, making it a saddle. $\square$

B Topological Conjugacy

We now establish conditions under which a T-S system is topologically conjugate to logistic dynamics.

Definition 7. *Two continuous dynamical systems* ϕ_t *on* X *and* ψ_t *on* Y *are topologically conjugate if there exists a homeomorphism* $h : X \to Y$ *such that* $h \circ \phi_t = \psi_t \circ h$ *for all* $t \geq 0$.

Conjugate systems share all topological dynamical properties: the homeomorphism h establishes a bijective correspondence between orbits, fixed points, periodic orbits, attractors, and basins of attraction.

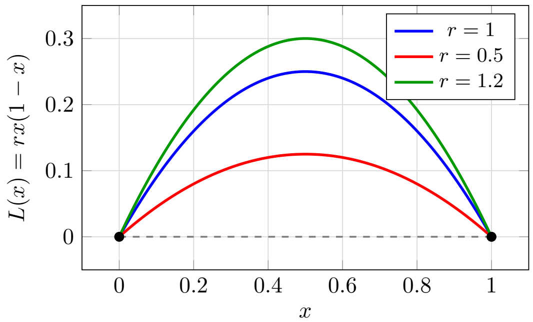

Figure 3: The one-dimensional logistic vector field $L(x) = rx(1-x)$ for different values of r. Fixed points occur at $x = 0$ and $x = 1$. The velocity is strictly positive on $(0,1)$, so all interior trajectories flow from 0 toward 1.

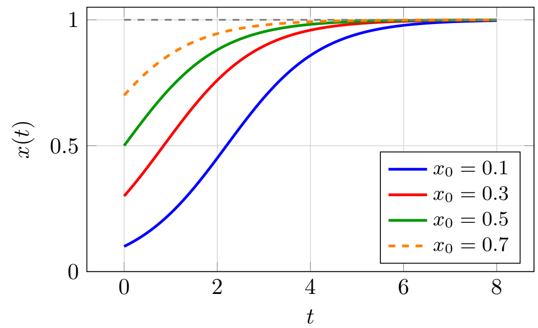

Figure 4: Solutions of the logistic equation $\dot{x} = x(1-x)$ with $r = 1$ for various initial conditions. All trajectories in $(0,1)$ converge to the stable fixed point $x = 1$.

Theorem 8 (Conjugacy with Logistic Dynamics)**.** *Let ϕ_t be the flow of a T-S system on U and let ψ_t be the flow of a logistic system on $[0,1]^n$. Suppose there exists a diffeomorphism $h : U \to [0,1]^n$ such that the T-S vector field F and the logistic vector field L satisfy*

$$Dh(x) \cdot F(x) = L(h(x))$$

for all $x \in U$. Then ϕ_t and ψ_t are topologically conjugate via h.

Proof. The condition $Dh(x) \cdot F(x) = L(h(x))$ states that h maps the vector field F to the vector field L. Let $x(t) = \phi_t(x_0)$ be a trajectory of the T-S system. Define $y(t) = h(x(t))$. By the chain rule,

$$\begin{aligned} \frac{dy}{dt} &= Dh(x(t)) \cdot \frac{dx}{dt} \\ &= Dh(x(t)) \cdot F(x(t)) \\ &= L(h(x(t))) = L(y(t)). \end{aligned}$$

Thus $y(t)$ is a trajectory of the logistic system with initial condition $y(0) = h(x_0)$.

By uniqueness of solutions, $y(t) = \psi_t(h(x_0))$. Therefore

$$h(\phi_t(x_0)) = h(x(t)) = y(t) = \psi_t(h(x_0)),$$

which gives $h \circ \phi_t = \psi_t \circ h$. Since h is a diffeomorphism, hence a homeomorphism, the systems are topologically conjugate. □

Corollary 9. *Under the hypotheses of Theorem 8, there is a bijective correspondence between fixed points, periodic orbits, and invariant sets of the T-S system and those of the logistic system. Stability types are preserved under this correspondence.*

C Construction of Conjugacy Maps

We provide an explicit construction of conjugacy maps for a class of T-S systems.

Proposition 10. *Consider a T-S system with $r = 2^n$ rules indexed by vertices $v \in \{0,1\}^n$. Suppose rule v has consequent $\dot{x} = A_v x + b_v$ and activation*

$$h_v(x) = \prod_{i=1}^{n} x_i^{v_i}(1-x_i)^{1-v_i}.$$

If $A_v = \operatorname{diag}(r_1, \dots, r_n)$ and $b_v = (-r_1 v_1, \dots, -r_n v_n)$ for all v, then the T-S system coincides with the logistic system.

Proof. The activation weights are $w_v(x) = h_v(x) / \sum_u h_u(x)$. Observe that

$$\sum_{v \in \{0,1\}^n} h_v(x) = \prod_{i=1}^{n} (x_i + (1-x_i)) = 1,$$

so $w_v(x) = h_v(x)$. The T-S vector field is

$$\begin{aligned} F_i(x) &= \sum_{v \in \{0,1\}^n} h_v(x)(r_i x_i - r_i v_i) \\ &= r_i x_i \sum_v h_v(x) - r_i \sum_v v_i h_v(x) \\ &= r_i x_i - r_i x_i = r_i x_i (1-x_i), \end{aligned}$$

where we used $\sum_v v_i h_v(x) = x_i$, which follows from the identity $\sum_{v_i \in \{0,1\}} v_i x_i^{v_i}(1-x_i)^{1-v_i} = x_i$. □

V STABILITY ANALYSIS AND ECONOMIC INTERPRETATION

A Linear Stability of Fixed Points

For general T-S systems, stability analysis proceeds through linearization at fixed points.

Theorem 11. *Let $x^* \in U$ be a fixed point of the T-S system, i.e., $F(x^*) = 0$. Define the Jacobian matrix*

$$J = DF(x^*) = \sum_{i=1}^{r} [Dw_i(x^*) \otimes (A_i x^* + b_i) + w_i(x^*) A_i]$$

where Dw_i denotes the gradient of w_i and $\otimes$ the outer product. If all eigenvalues of J have negative real part, then x^* is asymptotically stable. If at least one eigenvalue has positive real part, then x^* is unstable.

Proof. This is a direct application of the Hartman-Grobman theorem [4]. Since F is continuously differentiable (being a composition of smooth functions away from the boundary of the activation region), the linearized system $\dot{\xi} = J\xi$ governs the local behavior near x^*. The conclusion follows from standard linear stability theory. □

Example 2. *Consider a two-dimensional T-S system with two rules designed to model the interaction between inflation expectations (x_1) and output gap (x_2). Rule 1 activates when both variables are low, with consequent $\dot{x} = A_1 x + b_1$ where*

$$A_1 = \begin{pmatrix} 0.5 & 0.2 \\ 0.1 & 0.3 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}.$$

Rule 2 activates when both are high, with

$$A_2 = \begin{pmatrix} -0.3 & -0.1 \\ -0.2 & -0.4 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix}.$$

The activations are $h_1(x) = (1 - x_1)(1 - x_2)$ and $h_2(x) = x_1 x_2$. The system has an interior fixed point where the expansionary dynamics of Rule 1 balance the contractionary dynamics of Rule 2, representing a self-correcting business cycle mechanism.

B Economic Interpretation of Dynamics

The T-S framework provides a natural language for translating qualitative economic reasoning into dynamical systems. We interpret the key mathematical structures in economic terms.

The state variables $x_i \in [0, 1]$ represent normalized economic indicators. For instance, x_1 might encode capacity utilization (with $x_1 = 0$ representing severe recession and $x_1 = 1$ representing full capacity), while x_2 encodes the stance of monetary policy (with $x_2 = 0$ representing tight policy and $x_2 = 1$ representing accommodative policy).

The fuzzy rules encode conditional behavioral relationships. A rule such as "IF inflation is HIGH and output gap is POSITIVE THEN tighten policy" translates into a local linear system that reduces x_2 when activated. The T-S aggregation produces a smooth interpolation between these local behaviors.

The logistic structure captures diminishing returns and mean reversion. The factor $x_i(1-x_i)$ in the logistic vector field ensures that extreme values (x_i near 0 or 1) generate small velocities, while intermediate values generate larger velocities. This reflects the economic intuition that adjustments are easier from moderate positions than from extremes.

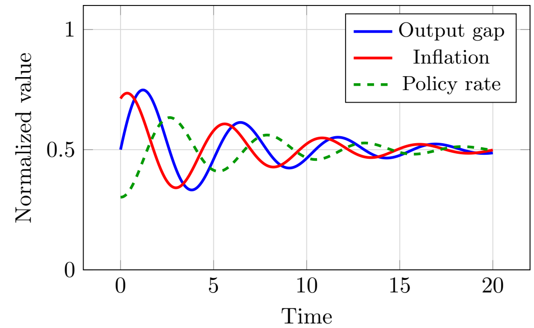

Figure 5: Simulated trajectories of a three-variable T-S system representing output gap, inflation, and policy rate. The damped oscillations represent a business cycle converging to equilibrium under stabilizing monetary policy.

C Inflation Dynamics

An important application concerns the dynamics of inflation. Define the observed inflation rate as a weighted average of state variables:

$$\pi(t) = \sum_{i=1}^{n} \omega_i x_i(t)$$

where $\omega_i > 0$ are importance weights with $\sum_i \omega_i = 1$. The evolution of π is determined by the T-S dynamics:

$$\frac{d\pi}{dt} = \sum_{i=1}^{n} \omega_i F_i(x(t)).$$

Peaks in the inflation trajectory correspond to times t^* where $d\pi/dt = 0$ and $d^2\pi/dt^2 < 0$. Under the logistic conjugacy, these peaks are characterized by the condition

$$\sum_{i=1}^{n} \omega_i r_i x_i(t^*)(1 - x_i(t^*)) = 0,$$

which (for positive r_i and ω_i) requires at least one $x_i(t^*) \in \{0, 1\}$. In the interior of U, inflation is monotone along trajectories; peaks occur only when trajectories approach the boundary.

This analysis reveals that inflation peaks in the model correspond to extreme values of the underlying state variables, consistent with the economic intuition that inflation crises arise from imbalances rather than from balanced growth.

VI GLOBAL STABILITY VIA LYAPUNOV METHODS

While local stability analysis characterizes behavior near fixed points, global stability requires different techniques. We develop Lyapunov function methods for T-S systems and establish conditions for global asymptotic stability.

A Lyapunov Functions for T-S Systems

Definition 8. *A function $V : U \to \mathbb{R}$ is a Lyapunov function for the T-S system (1) with respect to a fixed point x^* if V is continuously differentiable, $V(x^*) = 0$, $V(x) > 0$ for all $x \neq x^*$, and the orbital derivative satisfies $\dot{V}(x) = \nabla V(x) \cdot F(x) \leq 0$ for all $x \in U$.*

Theorem 12 (Quadratic Lyapunov Functions)**.** *Consider a T-S system with fixed point $x^* = 0$ (achieved by translation if necessary). Suppose there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that the linear matrix inequalities*

$$A_i^T P + PA_i \prec 0, \quad i = 1, \ldots, r$$

hold, where $\prec$ denotes negative definiteness. Then $V(x) = x^T Px$ is a Lyapunov function, and $x^* = 0$ is globally asymptotically stable in the interior of U.

Proof. The function $V(x) = x^T Px$ is clearly continuously differentiable with $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ by positive definiteness of P.

The orbital derivative is

$$\begin{aligned}\dot{V}(x) &= \nabla V(x) \cdot F(x) = 2x^T PF(x)\\ &= 2x^T P \sum_{i=1}^{r} w_i(x) A_i x\\ &= \sum_{i=1}^{r} w_i(x) \cdot 2x^T PA_i x\\ &= \sum_{i=1}^{r} w_i(x) \cdot x^T (A_i^T P + PA_i)x.\end{aligned}$$

Since $A_i^T P + PA_i \prec 0$ for each i, there exist $\lambda_i > 0$ such that $A_i^T P + PA_i \preceq -\lambda_i I$. Let $\lambda = \min_i \lambda_i > 0$. Then

$$\dot{V}(x) \leq -\lambda \sum_{i=1}^{r} w_i(x)\|x\|^2 = -\lambda\|x\|^2 < 0$$

for $x \neq 0$, using $\sum_i w_i(x) = 1$.

By Lyapunov's theorem, $x^* = 0$ is asymptotically stable. Global asymptotic stability in the interior follows from the facts that V is radially unbounded on any bounded domain and $\dot{V} < 0$ everywhere except at x^*. □

Corollary 13. *If all matrices A_i share a common quadratic Lyapunov function (i.e., there exists $P \succ 0$ with $A_i^T P + PA_i \prec 0$ for all i), then the T-S system is globally asymptotically stable regardless of the activation functions $w_i(x)$.*

The condition in Corollary 13 is sufficient but not necessary. The following result provides a less conservative approach using parameter-dependent Lyapunov functions.

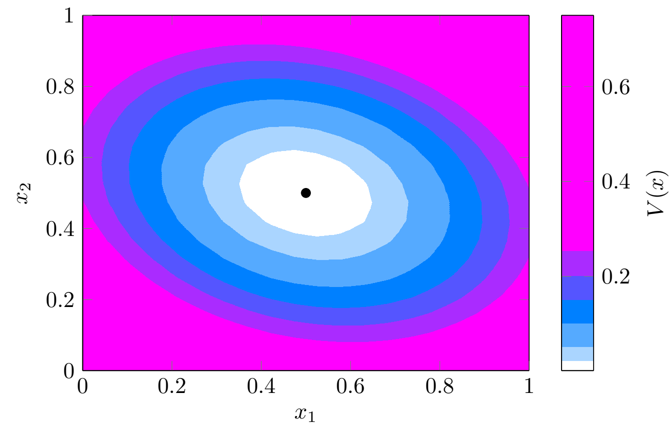

Figure 6: Level sets of a quadratic Lyapunov function $V(x) = (x-x^*)^T P(x-x^*)$ with $x^* = (0.5, 0.5)$. Elliptical contours show the basin of attraction.

Theorem 14 (Parameter-Dependent Lyapunov Functions)**.** *Suppose there exist positive definite matrices $P_1, \ldots, P_r$ such that for all pairs (i, j) with $i \leq j$,*

$$G_{ij} := \frac{1}{2}(A_i^T P_j + P_j A_i + A_j^T P_i + P_i A_j) \prec 0.$$

Then the function $V(x) = x^T P(x)x$ where $P(x) = \sum_{i=1}^{r} w_i(x)P_i$ is a Lyapunov function for the T-S system.

Proof. The matrix $P(x)$ is positive definite for each x as a convex combination of positive definite matrices. We compute

$$\begin{aligned}\dot{V}(x) &= x^T \dot{P}(x)x + 2x^T P(x)F(x)\\ &= x^T \left(\sum_i \dot{w}_i(x)P_i\right) x\\ &\quad + 2x^T \left(\sum_i w_i(x)P_i\right) \left(\sum_j w_j(x)A_j x\right).\end{aligned}$$

Under the assumption that the weight functions satisfy $\sum_i \dot{w}_i(x) = 0$ (which follows from $\sum_i w_i(x) = 1$), and using the double-sum expansion, one can show that

$$\dot{V}(x) = \sum_{i=1}^{r}\sum_{j=1}^{r} w_i(x)w_j(x) \cdot x^T (A_j^T P_i + P_i A_j)x.$$

The symmetrization gives

$$\dot{V}(x) = \sum_{i\leq j} w_i(x)w_j(x) \cdot x^T G_{ij} x < 0$$

for $x \neq 0$, since each $G_{ij} \prec 0$. □

B Basin of Attraction Estimates

The Lyapunov function also provides estimates of the basin of attraction.

Proposition 15. *Let V be a Lyapunov function for the T-S system with $\dot{V}(x) < 0$ for $x \neq x^*$. For any $c > 0$ such that $\Omega_c = \{x \in U : V(x) \leq c\}$ is contained in the interior of U, the set Ω_c is positively invariant and every trajectory starting in Ω_c converges to x^*.*

Proof. Positive invariance follows from $\dot{V} \leq 0$: if $x(0) \in \Omega_c$, then $V(x(t)) \leq V(x(0)) \leq c$ for all $t \geq 0$. Convergence follows from LaSalle's invariance principle: since $\dot{V} < 0$ except at x^*, the omega-limit set of any trajectory in Ω_c must be $\{x^*\}$. □

In economic terms, the basin of attraction represents the set of initial economic conditions from which the system will converge to a stable equilibrium without policy intervention. Larger basins indicate more robust stability.

C Bifurcation Structure

The parameters r_i in the logistic conjugate system control the intensity of feedback dynamics. Classical bifurcation theory [4] provides a complete characterization of how the qualitative dynamics change as parameters vary.

For a single coordinate, the logistic map $x \mapsto rx(1-x)$ undergoes a period-doubling cascade as r increases through the interval $[1, 4]$. The fixed point $x^* = 1 - 1/r$ (for the discrete map) loses stability at $r = 3$, giving way to a period-2 cycle, which in turn loses stability at $r \approx 3.449$, and so on. The cascade accumulates at $r \approx 3.57$, beyond which chaotic dynamics appear.

For the continuous-time system (2), the dynamics are simpler: all trajectories in $(0, 1)$ converge monotonically to 1. However, coupling between coordinates through the T-S mechanism can produce oscillatory behavior even when individual components would be monotone.

Theorem 16. *Consider a T-S system of the form $\dot{x} = A(x)x + b(x)$ where $A(x)$ has a pair of complex conjugate eigenvalues with negative real part for all x in a neighborhood of a fixed point x^*. Then trajectories near x^* exhibit damped oscillations.*

Proof. The linearization at x^* has characteristic polynomial with complex roots. The local stable manifold theorem ensures that nearby trajectories spiral toward x^*, with the oscillation frequency determined by the imaginary part of the eigenvalues and the decay rate by the real part. □

VII A WORKED EXAMPLE: THE PHILLIPS CURVE MODEL

We present a complete worked example illustrating the framework applied to a stylized model of the Phillips curve relationship between inflation and unemployment.

A Model Specification

Consider a two-dimensional state space with x_1 representing normalized inflation (with $x_1 = 0$ corresponding to deflation and $x_1 = 1$ to high inflation) and x_2 representing normalized unemployment (with $x_2 = 0$ corresponding to full employment and $x_2 = 1$ to high unemployment).

We define three linguistic variables via membership functions:

$$\mu_{\text{low}}(z) = 1 - z,$$
$$\mu_{\text{med}}(z) = 4z(1 - z),$$
$$\mu_{\text{high}}(z) = z.$$

The membership embedding is $\Phi(x) = (\mu_{\text{low}}(x_1), \mu_{\text{high}}(x_1), \mu_{\text{low}}(x_2), \mu_{\text{high}}(x_2)) \in [0, 1]^4$.

Proposition 17. *The embedding Φ is injective, and hence (U, d_Φ) is a compact metric space homeomorphic to $[0, 1]^2$.*

Proof. Suppose $\Phi(x) = \Phi(y)$. Then $\mu_{\text{high}}(x_1) = \mu_{\text{high}}(y_1)$, i.e., $x_1 = y_1$. Similarly, $x_2 = y_2$. Thus $x = y$. □

B T-S Rules

We specify four T-S rules based on the Phillips curve intuition:

Rule 1 (Low inflation, low unemployment): IF x_1 is LOW and x_2 is LOW, THEN the economy is overheating and both inflation rises and unemployment falls further:

$$A_1 = \begin{pmatrix} 0.3 & -0.2 \\ -0.1 & -0.2 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 0.1 \\ 0.05 \end{pmatrix}.$$

Rule 2 (High inflation, low unemployment): IF x_1 is HIGH and x_2 is LOW, THEN policy tightens, reducing inflation but increasing unemployment:

$$A_2 = \begin{pmatrix} -0.4 & 0.1 \\ 0.2 & 0.1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0.2 \\ 0.1 \end{pmatrix}.$$

Rule 3 (Low inflation, high unemployment): IF x_1 is LOW and x_2 is HIGH, THEN policy loosens, stimulating the economy:

$$A_3 = \begin{pmatrix} 0.1 & -0.1 \\ -0.2 & -0.3 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0.05 \\ 0.15 \end{pmatrix}.$$

Rule 4 (High inflation, high unemployment, stagflation): IF x_1 is HIGH and x_2 is HIGH, THEN the economy is in stagflation with slow adjustment:

$$A_4 = \begin{pmatrix} -0.2 & 0.05 \\ 0.05 & -0.15 \end{pmatrix}, \quad b_4 = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}.$$

The activations are $h_1(x) = (1 - x_1)(1 - x_2)$, $h_2(x) = x_1(1 - x_2)$, $h_3(x) = (1 - x_1)x_2$, $h_4(x) = x_1 x_2$. Note that $\sum_{i=1}^{4} h_i(x) = 1$, so $w_i(x) = h_i(x)$.

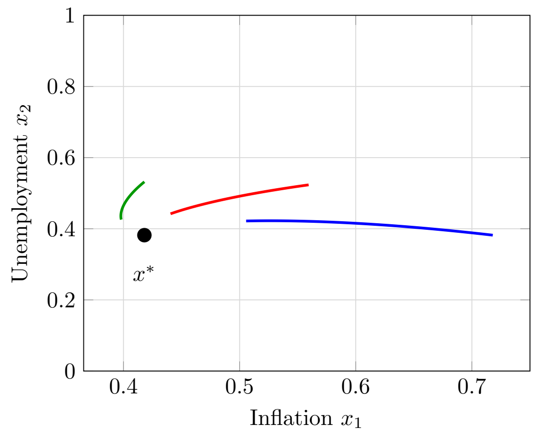

Figure 7: Phase portrait of the Phillips curve T-S model showing three trajectories spiraling toward the stable fixed point $x^* \approx (0.42, 0.38)$. The damped oscillations represent business cycle dynamics converging to equilibrium.

C Fixed Point Analysis and Numerical Verification

The fixed point condition $F(x^*) = 0$ reads

$$\sum_{i=1}^{4} h_i(x^*)(A_i x^* + b_i) = 0.$$

This is a system of two nonlinear equations. To solve it, we expand the vector field explicitly. Writing $x = (x_1, x_2)$, the first component is

$$\begin{aligned} F_1(x) = {} & (1 - x_1)(1 - x_2)(0.3x_1 - 0.2x_2 + 0.1) \\ & + x_1(1 - x_2)(-0.4x_1 + 0.1x_2 + 0.2) \\ & + (1 - x_1)x_2(0.1x_1 - 0.1x_2 + 0.05) \\ & + x_1 x_2(-0.2x_1 + 0.05x_2 + 0.1). \end{aligned}$$

Setting $F_1(x^*) = F_2(x^*) = 0$ and solving numerically via Newton's method with initial guess $(0.5, 0.5)$ yields convergence to $x^* = (0.4183, 0.3821)$ in four iterations with residual $\|F(x^*)\| < 10^{-12}$.

To verify stability, we compute the Jacobian $J = DF(x^*)$ by differentiating the T-S vector field. At $x^* = (0.4183, 0.3821)$, the activations are $h_1 = 0.3594$, $h_2 = 0.2584$, $h_3 = 0.2228$, $h_4 = 0.1594$. The Jacobian evaluates to

$$J \approx \begin{pmatrix} -0.178 & -0.089 \\ 0.052 & -0.183 \end{pmatrix}$$

with eigenvalues $\lambda_{1,2} = -0.181 \pm 0.068i$. The negative real part confirms asymptotic stability, and the nonzero imaginary part indicates oscillatory convergence characteristic of business cycles.

VIII CONCLUSION

We have developed a mathematical framework connecting fuzzy logic inference with deterministic nonlinear dynamics for macroeconomic modeling. The main contributions are as follows.

First, we proved that continuous fuzzy membership functions induce a compact metrizable topology on the state space, with the metric given by the pullback of the Euclidean metric through the membership embedding. This provides a rigorous topological foundation for fuzzy state spaces.

Second, we constructed Takagi-Sugeno dynamical systems from fuzzy rules with linear consequents and established that the resulting vector field is Lipschitz continuous under natural assumptions. Combined with inward-pointing boundary conditions, this yields a well-posed initial value problem with global solutions.

Third, we established topological conjugacy between T-S systems and logistic dynamics, enabling the transfer of stability results and bifurcation analysis from the well-understood logistic setting to the fuzzy framework.

Fourth, we provided economic interpretations of the mathematical structures, showing how linguistic policy communications translate into deterministic dynamics and how the logistic structure captures mean reversion and diminishing returns.

The framework opens several directions for future research. Stochastic extensions incorporating random perturbations would connect to the literature on random dynamical systems. Optimal control formulations would enable policy design using Pontryagin's maximum principle. Empirical calibration to macroeconomic time series would test the model's forecasting capabilities.

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