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Air-Entrapment Issues During Draining Processes in Water Supply Systems: Towards an Analytical Solution for pressure in Inclined Pipelines

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***Abstract*—This study examines the challenges associated with air entrapment during the draining process in water supply systems, focusing on developing an analytical solution for the dynamics of the pressure in inclined pipelines. Air pockets trapped in water columns can lead to various operational issues, such as water hammer, flow reduction, pressure oscillations, and potential structural damage. This research builds on previous numerical models that describe air-water interactions in a single inclined pipe with a sealed upper end. The model was initially formulated using a system of three equations: two ordinary differential equations (ODEs) and one algebraic equation. The primary objective of this study is to derive an analytical solution for the case where the air pocket remains closed. To achieve this, the initial model is reduced to a system of two first-order nonlinear ODEs, allowing for an analysis of the existence and uniqueness of solutions. It is then further transformed into a second-order nonlinear ODE to facilitate an intuitive examination of its oscillatory behavior. A numerical validation of this model confirms its accuracy in predicting the physical system's behavior. Additionally, through a variable transformation, the second-order ODE is converted into a first-order linear ODE, potentially simplifying the derivation of an explicit analytical solution. This research extends the understanding of transient hydraulic dynamics during the draining process in pipelines with air-water interaction. The findings provide an analytical framework that complements previous numerical solutions, offering valuable insights for optimizing hydraulic system design and performance.**

***Index Terms*—Pipeline drainage, Water-air interaction, Pressure oscillations, Nonlinear differential equations, Air pocket behavior.**

I. INTRODUCTION

In hydraulic water supply systems, the filling and emptying processes of pipelines can cause several inconveniences when air pockets get trapped between the water columns. Some of the main complications include: (1) Water hammer; (2) Reduction in effective flow; (3) Pressure oscillations; (4) Risk of partial vacuum; (5) Damage to equipments; (6) Noise and vibration; (7) Difficulties in controlling emptying. Mitigating these problems requires a proper system design, including the incorporation of air valves and operating strategies to handle hydraulic transients and ensure an efficient purge of trapped air during the filling and emptying processes of the system [1].

The emptying of a water column in pipes with air-water interaction is a complex phenomenon that involves highly non-linear transient dynamics. Vicente S. et al. [2] developed a model that describes the emptying of a water column by the gravity force in an inclined pipe with an air pocket at the upper end. The model is based on a system of three equations (two ordinary differential equations (ODE) and one algebraic) which describes the behavior of the water-air interaction in two specific cases: (1) when the air pocket is closed, and (2) when the air pocket has a valve to regulate the air release. This model was validated by numerical solutions, and these solutions compared with experimental data obtained under controlled conditions. This study served as a fundamental basis for understanding the physical and mathematical factors associated with these systems. However, an analytical solution is yet to be found.

In this novel work, progress is made towards obtaining an analytical solution for case (1), where the air pocket remains closed. To this end, the proposed system is first reduced to a system of two first-order nonlinear differential equations. This approach allows a rigorous analysis of the existence and uniqueness of the solutions. Then, the system is reduced to a second-order nonlinear ODE, granting an intuitive analysis of the solutio's oscillatory behavior. This second-order ODE is solved numerically, corroborating that the proposed ODE models the physical system with the parameters determined a priori. Furthermore, through a special change of variables, the second-order ODE is transformed into a first-order linear ODE, which could simplify the process of proposing an analytical solution.

This work extends the knowledge developed in the previous study, providing analytical tools that complement the numerical solutions previously obtained. In doing so, we aim not only to deepen our understanding of the mechanisms involved in pipe emptying, but also to offer an analytical framework that can be used for the design and optimization of similar hydraulic systems.

II. Mathematical Model

In this section we introduce once more the mathematical model to analyze the emptying process in a single pipe proposed in [2]. Fig. 1 shows the diagram of an air pocket trapped in a single pipe while the water is emptied.

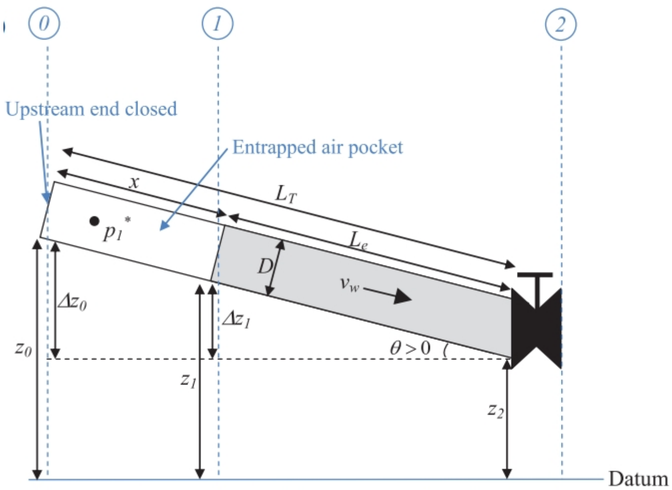

Fig. 1. Graphical scheme of an air pocket trapped into a single pipe while water is being drained (from [2]).

The emptying process in a single pipe, with its upper end closed involves an air pocket contained inside the pipe. For the analysis of this phenomenon, the following assumptions were made:

- Water's dynamics is modeled as if it were a rigid (incompressible) column.
- The slope, the diameter and the roughness of the pipe do not change during the experiment. For this reason, they are considered constant.
- A constant friction factor is set such that it represents the losses by Darcy-Weisbach equation [3]–[7].
- Although the upper end of the pipe remains closed, a polytropic evolution of the air pocket is considered that allows the pipe to be emptied [7]. For this purpose, a polytropic coefficient is introduced to model the dynamics of the trapped air. This coefficient takes values from 0.0 (isothermal) to 1.4 (adiabatic). In the pouring process the experiment is performed in an intermediate situation and the polytropic coefficient takes values between 0.0 and 1.4.
- A valve is installed at the lower end of the pipe to regulate the water drainage.
- Although the air-water interface should be actually horizontal, it is assumed to be a well-defined cross section which can be applied to individual pipes with small diameters and hydraulic slopes such that no free surface flow arises [5].
- The pipe can withstand dangerous drops in sub-atmospheric pressure during transient phenomena.

With the above assumptions, the phenomenon is modeled as:

- **Emptying column**. The mass velocity oscillation equation of water for a pouring column (rigid column method) is

$$\frac{dv_w}{dt} = \frac{p_1^* - p_{atm}^*}{\rho_w L_e} + g\sin(\theta) - \frac{f v_w |v_w|}{2D} - \frac{R_v g A^2 v_w |v_w|}{L_e}, \quad (1)$$

  where v_w = velocity of the water column, p_1^* = absolute pressure of the air pocket, p_{atm}^* = atmospheric pressure, ρ_w = water density, L_e = length of the emptying column, g = gravity acceleration, θ = pipe slope (rad), f = Darcy-Weisbach friction factor, D = pipe inner diameter, A = cross-section area of the pipe, R_v = resistance coefficient and Q_w = water flow. Minor losses through the valve are estimated using the formula $h_m = R_v Q_w^2$.

- **Gravity realted term**
  The gravity related term (z_1/L_e) present in (1) is constant and can be modeled as:

$$\sin(\theta) = \Delta z_1 / L_e,$$

  where, Δz_1 is the elevation difference (see Fig. 1).

- **Air-water interface.** The position of the emptying column interface is formulated as

$$\frac{dL_e}{dt} = -v_w, \quad (2)$$

  where $L_{e,0=}$ is the initial value of L_e.

- **Air pocket.** The air pocket is represented as

$$p_1^* V_a^k = p_{1,0}^* V_{a,0}^k \quad or \quad p_1^* x^k = p_{1,0}^* x_0^k, \quad (3)$$

  where V_a = air volume, $V_{a,0}$ = initial air volume, $p_{1,0}^*$ = initial value for $p*_1$, k= polytropic coefficient, x = length of the trapped air pocket and x_0 = initial value of the length x.

- **Initial and boundary conditions**. Initially ($t = 0$), the system is assumed at rest . The initial conditions are given by $v_w(0) = 0$, $L_{e,0} = L_T - x_0$ y $p_{1,0}^* = p_{atm}^*$.
  The upstream boundary condition is given as $p_{1,0}^*$ (initial condition for the air pocket). Downstream, the boundary condition is given by p_{atm}^* (free water release to the atmosphere).

In summary, the system (1)–(3) describes the process. This system, together with the corresponding initial and boundary conditions, can be solved for (v_w, L_e and p_1^*). In brief, the resulting system is:

$$\frac{dv_w}{dt} = \frac{p_1^* - p_{atm}^*}{\rho_w L_e} + g\sin(\theta) - \frac{f v_w |v_w|}{2D} - \frac{R_v g A^2 v_w |v_w|}{L_e} \quad (1)$$

$$\frac{dL_e}{dt} = -v_w \quad (2)$$

$$p_1^* = \frac{p_{1,0}^* (L_T - L_e)^k}{(L_T - L_{e,0})^k}; \quad (3)$$

with initial conditions

$$v_w = 0, \quad L_{e,0} = L_T - x_0, \quad \text{and} \quad p^*_{1,0} = p^*_{atm} \tag{4}$$

III. Existence and uniqueness of the system solution

In this section we will show that the system (1)-(3) subject to the given initial conditions (4) has a unique solution on certain intervals. To this end, the system is reduced to a set of nonlinear differential equations as follows:

Let $L := L_e(t)$ and $v := v_w(t)$ variables that depend on time t. Combining (1), (2) and (3), the system reduces to a two-dimensional system of nonlinear differential equations:

$$\frac{dL}{dt} = -\, v \tag{5}$$
$$\frac{dv}{dt} = \frac{a}{(L_T - L)^k L} + b - v|v|\left(c + \frac{d}{L}\right) - \frac{p^*_{atm}}{\rho_\omega}\frac{1}{L}; \tag{6}$$

with initial conditions

$$L(0) = L_T - x_0 \quad \text{and} \quad v(0) = 0; \tag{7}$$

where

$a = \frac{p^*_{atm}(x_0)^k}{\rho_w}$, $b = g\sin(\theta)$, $c = \frac{f}{2D}$, and $d = R_v g A^2$.

The system (5)-(7) can be written more compactly in vector notation as

$$\begin{pmatrix} \dot{L} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} P(L,v) \\ Q(L,v) \end{pmatrix} \tag{8}$$

with

$$\begin{pmatrix} \dot{L}(0) \\ \dot{v}(0) \end{pmatrix} = \begin{pmatrix} L_T - x_0 \\ 0 \end{pmatrix} \tag{9}$$

where P, Q: $(0, L_T) \times \mathbb{R} \to \mathbb{R}$, are funtions defined by:

$$P(L,v) := -\, v$$
$$Q(L,v) := \frac{a}{(L_T - L)^k L} + b - v|v|\left(c + \frac{d}{L}\right) - \frac{p^*_{atm}}{\rho_\omega}\frac{1}{L}$$

Note that the funtions P and Q do not explicitly depend on the variable t (time). Therefore (8) is nonlinear autonomous systems of differential equations with initial values (9), whose solutions $(L(t), v(t))$ are parameterized curves in the phase plane (L, v) called orbits. By direct calculations we have that:

$$\frac{\partial Q}{\partial L} = \frac{1}{L^2}\left[\frac{p^*_{atm}}{\rho_\omega} + v|v|d - \frac{a}{(L_T - L)^K}\right] - \frac{ak}{L(L_T - L)^{k+1}}$$
$$\frac{\partial Q}{\partial v} = \left(c + \frac{d}{L}\right) 2|v|.$$
$$\frac{\partial P}{\partial L} = 0 \qquad \text{and} \qquad \frac{\partial P}{\partial v} = -1.$$

Note that P, Q, $\frac{\partial P}{\partial L}$, $\frac{\partial P}{\partial v}$, $\frac{\partial Q}{\partial L}$ and $\frac{\partial Q}{\partial v}$ are continuou in open connected set $(0, L_T) \times \mathbb{R}$.

This is, because according to the parameters of the system, L is never zero and L_T is never equal to L during the emptying process because it starts ($t = 0$) from $L_T - x_0$ (see Fig.1). Then, these conditions on P and Q guarantee that for $X_0 := (L_T - x_0, 0) \in (0, L_T) \times \mathbb{R}$, the initial value problem has a solution $X(t) := (L(t), v(t))$ on some time interval $[0, \tau)$, for a given $\tau > 0$ and the solution is unique in that interval (see [8], Chap.6)

IV. An analytical approach to the solution

Instead of dealing with problem (1)-(4), we now proceed to derive a more simplified initial value problem (IVP) for L with initial conditions $L(0) = L_T - x_0$ and $\frac{dL}{dt}(0) = 0$. Here, we propose a reduced model for the process of emptying a pipe under the conditions given in Section II. This model consists of a nonlinear second order ODE with given initial conditions. Then, we verify numerically that the reduced model matches the original. Finally, by means of a appropriate substitution we reduce the problem to one consisting of a first order ODE, which might simplify the solution process.

A. Reduction of the System to a Second Order ODE

Combining (5)-(6), we obtain the following second-order nonlinear ordinary differential equation in L:

$$-L\frac{d^2L}{dt^2} = \frac{a}{(L_T - L)^k} + b\,L + c\,\frac{dL}{dt}\left|\frac{dL}{dt}\right| L + d\,\frac{dL}{dt}\left|\frac{dL}{dt}\right| - \frac{p^*_{atm}}{\rho_w} \tag{10}$$

Subject to the following initial conditions

$$L(0) = L_T - x_0 \quad \text{and} \quad \frac{dL}{dt}(0) = 0. \tag{11}$$

B. Numerical verification of initial value problem

In this section a numerical solution is performed for both the original system (1)-(4) and the reduced system (10)-(11), in order to determine if they are equivalent.

Initial value problem (IVP) (10) with initial conditions $L(0) = L_T - x_0$ and $\frac{dL}{dt}(0) = 0$ can be approximated by a second order finite difference scheme as follows (for more details see [9]):

$$\begin{aligned} -L_i\frac{L_{i+1} - 2\,L_i + L_{i-1}}{(\Delta t)^2} &= \frac{a}{(L_T - L_i)^k} + b\,L_i \\ &+ c\,L_i\frac{L_i - L_{i-1}}{\Delta t_i}\left|\frac{L_i - L_{i-1}}{\Delta t}\right| \\ &+ d\frac{L_i - L_{i-1}}{\Delta t}\left|\frac{L_i - L_{i-1}}{\Delta t}\right| - \frac{p^*_{atm}}{\rho_w} \quad (12) \\ L_0 &= L_T - x_0 \quad (13) \\ \frac{L_1 - L_0}{\Delta t} &= 0, \quad (14) \end{aligned}$$

defined on a finite time interval $[t_I, t_F]$ which is split into N subintervals $[t_{i-1}, t_i]$ for $1 \leq i \leq N$. Here $\Delta t = t_i - t_{i-1}$ is a constant time stepsize and $L_i = L(t_i)$.

IVP 12 - 14 can be rewritten as

$$
\begin{aligned}
L_{i+1} &= -(\Delta t)^2 \left[\frac{a}{(L_T - L_i)^k \, L_i} + b + c \frac{L_i - L_{i-1}}{\Delta t_i} \left| \frac{L_i - L_{i-1}}{\Delta t} \right| \right. \\
&+ \left. \frac{d}{L_i} \frac{L_i - L_{i-1}}{\Delta t} \left| \frac{L_i - L_{i-1}}{\Delta t} \right| - \frac{p^*_{atm}}{\rho_w L_i} \right] - 2\, L_i + L_{i-1} \qquad (15) \\
L_0 &= L_T - x_0 \qquad (16) \\
L_1 &= L_0. \qquad (17)
\end{aligned}
$$

By applying a simple Forward Euler Iteration (Fwd Euler) iteration to IVP (15)-(17) an approximate solution is obtained for the unknown L at the nodes L_i.

Tests were performed over the time interval $[0, t_F]$ with $t_F \approx 2022$ s. The values for the parameters used are given in Table I.

TABLE I
VALUES OF PARAMETERS IN NUMERICAL TESTS.

Parameter	Value (Units)
L_T	$600\ (m)$
f	0.018
D	$0.35\ (m)$
R_v	$0.06\ \left(s^2\, m^{-6}\right)$
x_0	$200\ (m)$
k	1.2
g	$9.8\ (m\ s^{-2})$
ρ_w	$1000\ (Kg\, m^{-3})$
p^*_{atm}	$101325\ (Pa)$
θ	$\sin^{-1}(3/120)\ (rad)$
γ_w	$9.805\ (N\, m^{-3})$
A	$\frac{\pi D^2}{4}\ (m^2)$
a	$\frac{p^*_{atm}\, x_0^k}{\rho_w}\ (Pa\, m^{k+3}/\, Kg)$
b	$g \sin(\theta)\ (m\, s^{-2})$
c	$\frac{f}{2\, D}\ (m^{-1})$
d	$R_v\, g\, A^2\ (m^{-3})$

Fig. 2 - 5 show the evolution in time of the length L of the water column contained in the pipe, the velocity v of drainage of the water from the pipe, the pressure p_1^* of the air pocket in the pipe, and the water flow Q_w from the pipe, respectively. In addition, the phase plane (L, v) is illustrated in Fig. 5.

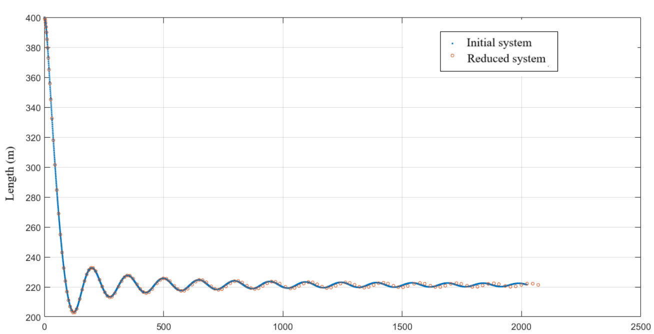

Fig. 2. Comparison of the numerical simulation of the length of the water body L (initial system vs. reduced system)

Fig. 2 - 5 demonstrate that the simplified model 10 matches accurately the dynamics of emptying a water pipe as described by the original model.

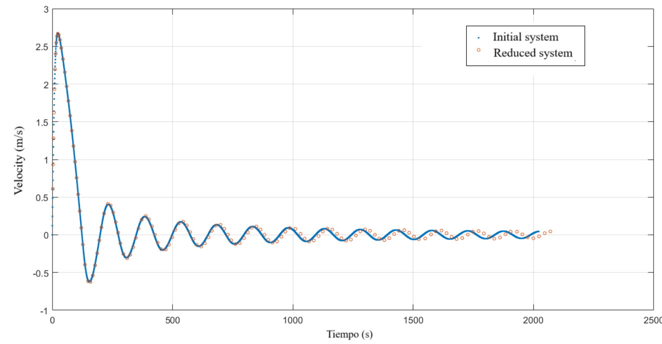

Fig. 3. Comparison of the numerical simulation of the velocity v at which the water drains from the pipe (initial system vs. reduced system)

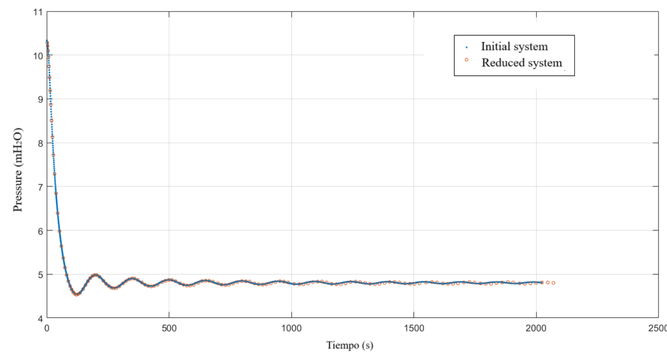

Fig. 4. Comparison of the numerical simulation of the pressure of the air pocket in the pipe p_1^* (initial system vs. reduced system)

C. Reduction of the Second Order ODE to a First Order ODE

In order to solve IVP (10) with $L(0) = L_T - x_0$ and $\frac{dL}{dt}(0) = 0$ we need to observe the involvement of the term $\left|\frac{dL}{dt}\right|$, which brings up two cases: $L'(t) \geq 0$ and $L'(t) < 0$. Fig. 3 evidences the oscillatory nature of $v = L'(t)$ about zero. This leads us to redefine the initial problem, given for all $t > 0$, in several sub-problems on time sub-intervals of the form $[t_k, t_{k+1}]$. These sub-intervals are determined by the changes in the sign of v, starting with $t_0 = 0$ (where $v(t_0) = 0$).
The value t_1 is set to $t_1 = \sup_t \{t > 0 : -v(t) = L'(t) \geq 0\}$.

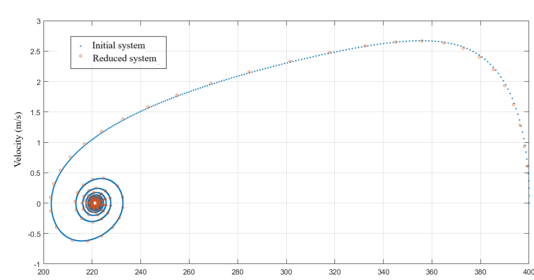

Fig. 5. Comparison of the numerical simulation of phase plane (L, v) (initial system vs. reduced system) .

With this condition, it is ensured that $L'(t)$ maintains the same sign for all $t \in (t_0, t_1)$, i.e.; $L'(t) > 0$.
Once a solution for IVP (10) with initial conditions $L(0) = L_T - x_0$ and $L'(0) = 0$ is obtained, the values $L(t_1)$ and $L'(t_1) = -v(t_1) = 0$ can be computed.
Since t_1 is the smallest value of t such that $t_1 > t_0$ and $L'(t_1) \geq 0$, then a new value t_2 can be set such that $t_2 = \sup_t \{t > t_1 : -v(t) = L'(t) \leq 0\}$. Thus, a new IVP can be defined using (10) and initial conditions $L(t_1)$ and $L'(t_1) = -v(t_1) = 0$.
We repeat this process solving the IVP given by (10) defined on $[t_k, t_{k+1}]$ and initial conditions $L(t_k)$ and $L'(t_k) = -v(k) = 0$ where t_{k+1} is set as

$$t_{k+1} = \sup_t \{t > t_k : -v(t) = L'(t) \geq 0 \text{ or } -v(t) = L'(t) \leq 0\}.$$

*1) **Case** $\frac{dL(t)}{dt} \geq 0$ **on the interval** $[t_k, t_{k+1}]$:* We first consider the case where $L'(t) \geq 0$ in $[t_k, t_{k+1}] \subset [0, \infty)$. Under these condition, (10) becomes

$$-L\frac{d^2L}{dt^2} = \frac{a}{(L_T - L)^k} + bL + c\left(\frac{dL}{dt}\right)^2 L + d\left(\frac{dL}{dt}\right)^2 - \frac{p^*_{atm}}{\rho_w}. \tag{18}$$

By using the substitution $u(L) := \frac{dL}{dt}$ the equation (18) becomes to the following linear first-order ODE for u^2:

$$\frac{d(u^2)}{dL} + P(L)u^2 = Q(L), \tag{19}$$

where

$$P(L) := 2c + \frac{2d}{L} \tag{20}$$

$$Q(L) := \frac{2p^*_{atm}}{\rho_w L} - 2b - \frac{2a}{(L_T - L)^k L} \tag{21}$$

Since $0 < x_0 < L < L_T$, the functions P and Q are continuous for all L evaluated in the interval $[t_k, t_{k+1}]$. So, (19) has a unique solution satisfying the initial condition

$$u_k := u(L(t_k)) = -v(t_k). \tag{22}$$

This solution is given by the formula: (see [10], Theorem 8.3):

$$u^2(L) = u_k^2 \cdot \exp(-A(L)) + \exp(-A(L)) \cdot B_1(L), \tag{23}$$

where

$$A(L) := \int_{L_{e,t_k}}^{L} P(s)\, ds \tag{24}$$

and,

$$B_1(L) := \int_{L_{e,t_k}}^{L} Q(s) \exp(A(s))\, ds. \tag{25}$$

Now, we explicitly calculate $\exp(A(L))$. Indeed,

$$\begin{aligned} A(L) &:= \int_{L_{e,t_k}}^{L} P(s)\, ds \\ &= \int_{L_{e,t_k}}^{L} \left(2c + \frac{2d}{s}\right) ds \\ &= 2c(L - L_{e,t_k}) + 2d \ln\left|\frac{L}{L_{e,t_k}}\right| \end{aligned}$$

Therefore,

$$\begin{aligned} \exp(A(L)) &= \exp(2c(L - L_{e,t_k})) \exp\left(\ln\left(\frac{L}{L_{e,t_k}}\right)^{2d}\right) \\ &= \left(\frac{L}{L_{e,t_k}}\right)^{2d} \exp(2c(L - L_{e,t_k})). \end{aligned}$$

So,

$$\exp(A(L)) = \left(\frac{L}{L_{e,t_1}}\right)^{2d} \exp(2c(L - L_{e,t_1})). \tag{26}$$

2) **Case** $\frac{dL(t)}{dt} \leq 0$ **on the interval** $[t_{k+1}, t_{k+2}]$**:** Where t_{k+1} is determinated by the solution on the interval $[t_k, t_{k+1}]$.
In this case, (10) takes the following form

$$-L\frac{d^2L}{dt^2} = \frac{a}{(L_T - L)^k} + bL - c\left(\frac{dL}{dt}\right)^2 L - d\left(\frac{dL}{dt}\right)^2 - \frac{p^*_{atm}}{\rho_w} \tag{27}$$

By using the substitution $u(L) := \frac{dL}{dt}$, (27) becomes to the following linear first-order ODE for u^2:

$$\frac{d(u^2)}{dL} - P(L)u^2 = Q(L), \tag{28}$$

where $P(L)$ and $Q(L)$ are the functions defined in (21).

By proceeding in a similar form as the case $\frac{dL}{dt} \geq 0$ and considering de initial condition

$$u_{k+1} := v(t_{k+1}) = -L'(t_{k+1}) = -u(L(t_{k+1})), \tag{29}$$

we have that (28) has a unique solution given by

$$u^2(L) = u_{k+1}^2 \cdot \exp(A(L)) + \exp(A(L)) \cdot B_2(L), \tag{30}$$

where

$$A(L) := \int_{L_{e,t_{k+1}}}^{L} P(s)\, ds \tag{31}$$

and,

$$B_2(L) := \int_{L_{e,t_{k+1}}}^{L} Q(s) \exp(-A(s))\, ds. \tag{32}$$

Remark. The calculation of $B_1(L)$ and $B_2(L)$ will be done in future work.

D. Solution of $V(t)$, $L(t)$ and p_1^* from u^2

a) Calculation of $v(t)$: We have that $u(L) = \frac{dL}{dt}$, then $u^2 := (u(L))^2 = (\frac{dL}{dt})^2$. So, $\frac{dL}{dt} = \pm\sqrt{u^2}$, and since $v(t) = -\frac{dL}{dt}$ we have:

$$v(t) = \mp\sqrt{u(L(t))^2}.$$

that is:

$$v(t) = \begin{cases} -\sqrt{u(L(t))^2}, & \text{if } \frac{dL}{dt} \geq 0 \\ \sqrt{u(L(t))^2}, & \text{if } \frac{dL}{dt} \leq 0 \end{cases} \tag{33}$$

b) Calculation of $L(t)$: Since $u(L(t)) = \frac{dL}{dt}$, we obtain

$$L(t) = \begin{cases} L(t_k) + \int_{t_k}^{t} \sqrt{(u(L(s)))^2}\, ds, & \text{if } \frac{dL}{dt} \geq 0 \\ L(t_{k+1}) - \int_{t_{k+1}}^{t} \sqrt{(u(L(s)))^2}\, ds, & \text{if } \frac{dL}{dt} \leq 0 \end{cases} \tag{34}$$

c) Calculation of p_1^*: Since we calculate L from u^2, then we can also compute the pressure as follows:

$$p_1^* = \frac{p_{1,0}^*(L_T - L_{e,0})^k}{(L_T - L)^k}. \tag{35}$$

In summary, the solution for 3×3 original system (1)-(4) is given by expressions (33), (34) and (35).

Finally, Fig. 6 - 8 show a graph of such solutions, where the integrals have been solved numerically. It can be seen that they agree with the numerical solution.

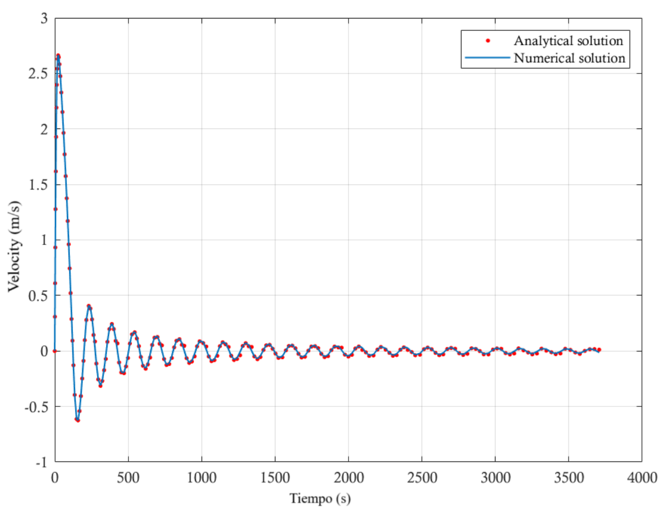

Fig. 6. Analytical solution for Velocity $v(t)$.

V. CONCLUSION AND FUTURE WORK

In this work, the problem of emptying a water column contained in a pipe with air-water interaction has been described and the mathematical model corresponding to this physical phenomenon has been presented. A strategy to simplify the model in order to determine a closed formula for the solution has been proposed. This strategy consisted of transforming the initial value problem into a linear problem in terms of a new variable. To validate the transformation performed, a numerical solution is found that coincides with the numerical solution of the original model; which implies that the simplified model is equivalent to the original model. Finally, the problem has been rewritten locally with respect to the time variable and a solution has been proposed in terms of integrals that have not yet been explicitly evaluated. The calculation of these integrals is a crucial step to obtain a closed form of the solution, which would facilitate its analysis and application in the study of the emptying of a water column in pipes with air-water interaction. In future work, the analytical resolution of these integrals can be addressed to analyze in depth their implications.

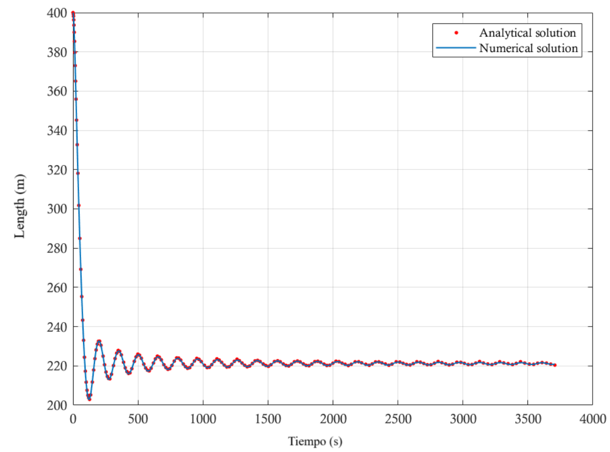

Fig. 7. Analytical solution for Length $L(t)$.

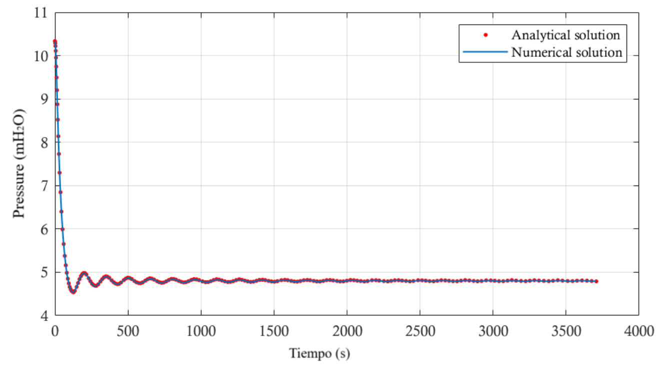

Fig. 8. Analytical solution for Pressure $p_1^*(t)$.

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