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Identities of incomplete elliptic integrals with application in electromagnetism

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Abstract*– In this article, a set of incomplete elliptic integrals is given to help engineering and physics students solve problems that are common in electromagnetism. First, a brief description of the elliptic integrals of the first kind and second kind is presented, followed by a list of identities of incomplete elliptic integrals involving sine and cosine functions, and finally, these identities are used to solve three physical systems: the first is the electric potential generated by a segment of a charged ring; the second is the magnetic field generated by a segment of a ring carrying a current; and the third is the magnetic field produced by an axially magnetized permanent magnet.

Keywords-- Elliptic integrals, electric potential, magnetic field, permanent magnet.

I. INTRODUCTION

Elliptic integrals are an essential tool in physics and engineering, as they are used in different areas of engineering such as electronics and aerospace engineering, among others. Such integrals allow to solve very complex problems such as the movement of celestial bodies in the case of classical mechanics [1] and in engineering we can find them in the calculation of the magnetic force between two magnets that form passive magnetic bearings [2], in the determination of areas and volumes, which is of great importance in civil and mechanical engineering, or in the solution of antenna design problems and in determining the impedance of electrical circuits in electric engineering [3].

The name "elliptic integrals" is derived from the fact that one of them is used to determine the arc of length of an ellipse, and the actual development of elliptic functions occurred between the 1650s and 1850s, when great mathematicians such as Fragnano, Euler, Lagrange, and Landen worked with them [4], [5] . The first man to carry out a more formal development on elliptic integrals was Legendre in his work "Traité des fonctions elliptiques" [6].

Due to the significance of these integrals in engineering and physics, the primary goal of this paper is to provide a set of identities involving incomplete elliptic integrals that can assist engineering and physics students in solving common electromagnetism problems. Particularly, the article provides a brief description of incomplete elliptic integrals of the first kind and the second kind (section II), ten identities of elliptic integrals involving sine and cosine functions that are frequently used in electromagnetism (section III), and three examples of how some of the identities are applied (section IV). The first is the electric potential produced by a segment of a charged ring, the second is the magnetic field produced by a segment of a ring carrying an electric current, and the third is the magnetic field produced by a segment of an annular permanent magnet magnetized in the axial direction.

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II. ELLIPTIC INTEGRALS

An ***elliptic integral*** is any integral that has the form

$$\int (a\cos\phi + b\cos\phi + c)\mathrm{d}\phi \tag{1}$$

Or

$$\int R\left(x, \sqrt{f(x)}\right)\mathrm{d}x\,, \tag{2}$$

where R is a rational function, f is a cubic or quartic polynomial of x with distinct roots [4], this is $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ ($a, b, c, d, e = constants$ and $a, b \neq 0$). Equation (1) is known as the Legendre's trigonometric form of elliptic integrals while equation (2) is known as Legendre's normal form.

Elliptic integrals are generally associated with three input arguments. In any of the Legendre forms, the first input argument is known as *amplitude*; in the trigonometric form it is ϕ, and in the normal form it is x. The second input argument used is m known as a *parameter* [7]. The third input argument is the modular angle α, which is defined as

$$\alpha = \sin^{-1}\sqrt{m}\,, \tag{3}$$

assuming that m is real and $0 \leq m \leq 1$

Finally, if an elliptic integral depends on two input arguments, this is known as an *incomplete elliptic integral*, but if it depends on a single argument, then it is known as a *complete elliptic integral*.

A. Elliptic integrals of the first kind

In Legendre's trigonometric form, the incomplete elliptic integral of the first kind is [8]

$$F(m,\phi)=\int_0^{\phi}\frac{d\theta}{\sqrt{1-m\sin^2\theta}},\qquad m<1. \qquad (4)$$

By changing the variable $z = \sin\theta$ the integral (4) is reduced to the normal form of Legendre

$$F(m,x)=\int_0^{x}\frac{dz}{\sqrt{(1-z^2)(1-mz^2)}},\qquad m<1, \qquad (5)$$

with $x=\sin\phi$. The complete elliptic integral of the first kind can be found by setting $\phi=\pi/2$ ($x=1$) in the equation (5), that is

$$F(m)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-m\sin^2\theta}}$$

$$=\int_0^{1}\frac{dz}{\sqrt{(1-z^2)(1-mz^2)}}, m<1. \qquad (6)$$

Using the binomial expansion theorem in the term $(1-m\sin^2\theta)^{-1/2}$ of equation (4), the incomplete elliptic integral of the first kind can be written as

$$F(m,\phi)=\phi+\frac{m}{2}\int_0^{\phi}sin^2\theta\,d\theta+\frac{3m^2}{8}\int_0^{\phi}sin^4\theta\,d\theta+\cdots$$
$$+\frac{1\cdot3\cdot5\cdots(2n-1)m^n}{2\cdot4\cdot6\cdots(2n)}\int_0^{\phi}\sin^{2n}\theta d\,\theta+\cdots \qquad (7)$$

B. *Elliptic integrals of the second kind*

The incomplete elliptic integral of the second kind is [8]

$$E(m,\phi)=\int_0^{\phi}\sqrt{1-m\sin^2\theta}\,d\theta,\qquad m<1 \qquad (8)$$

By changing the variable $z=\sin\theta$ the equation (8) is reduced to the normal form of Legendre is

$$E(m,x)=\int_0^{x}\sqrt{\frac{1-mz^2}{1-z^2}}dz,\qquad m<1, \qquad (9)$$

where $x=\sin\phi$. By setting $\phi=\pi/2$ ($x=1$), it is possible to find the value of the complete elliptic integral of the second kind

$$E(m)=\int_0^{\pi/2}\sqrt{1-m\sin^2\theta}\,d\theta=\int_0^{1}\sqrt{\frac{1-mz^2}{1-z^2}}dz, m<1 \qquad (10)$$

Similarly, as in equation (7), the binomial series can be used to expand $\sqrt{1-m\sin^2\theta}$ in equation (8), which yields to

$$E(m,\phi)=\phi-\frac{1}{2}m\int_0^{\phi}\sin^2\theta\,d\theta-\cdots$$
$$-\frac{1\cdot3\cdot5\cdots(2n-3)}{2\cdot4\cdot6\cdots(2n)}m^n\int_0^{\phi}\sin^{2n}\theta d\,\theta-\cdots \qquad (11)$$

III. SOME USEFUL IDENTITIES

In this section we present some identities that involve incomplete elliptic integrals that can be used in some problems in physics and engineering.

A. *Integrals that involve the sine function*

Using the expressions (4) and (8) we can find the followings identities that involve the sine function

$$\int_{\phi_1}^{\phi_2}\frac{\sin^2\theta\,d\theta}{(1-m\sin^2\theta)^{1/2}}=\frac{1}{m}\big(F(m,\phi_2)-F(m,\phi_1)$$
$$-E(m,\phi_2)+E(m,\phi_1)\big), \qquad (12)$$

$$\int_{\phi_1}^{\phi_2}\frac{d\theta}{(1-m\sin^2\theta)^{3/2}}=\frac{\big(E(m,\varphi_1)-E(m,\varphi_2)\big)}{1-m} \qquad (13)$$

$$\int_{\phi_1}^{\phi_2}\frac{\sin^2\theta\,d\theta}{(1-m\sin^2\theta)^{3/2}}=-\frac{\big(E(m,\varphi_2)-E(m,\varphi_1)\big)}{m(1-m)}$$
$$+\frac{1}{m}\big(F(m,\varphi_2)-F(m,\varphi_1)\big), \qquad (14)$$

Where

$$\varphi_i=\arccos\left[\frac{\sin\phi_i}{\sqrt{1+\frac{m}{1-m}\cos^2\phi_i}}\right]. \qquad (15)$$

If $\phi_1=0$ and $\phi_2=\pi/2$, the expressions (12) – (14) are simplified to the expressions shown in [9], which can be demonstrated using equations (15) and (83) – (86).

B. *Integrals that involve the cosine function*

With the help of equations (4), (8) and identities (12) - (14) we can find some identities of elliptic integrals that are related to the cosine function

$$\int_{\phi_1}^{\phi_2}\frac{d\theta}{(b-\cos\theta)^{1/2}}=\sqrt{2m}\big(F(m,\alpha_1)-F(m,\alpha_2)\big), \qquad (16)$$

$$\int_{\phi_1}^{\phi_2} \frac{d\theta}{(b+\cos\theta)^{1/2}} = \sqrt{2m}\big(F(m,\gamma_2) - F(m,\gamma_1)\big), \quad (17)$$

$$\int_{\phi_1}^{\phi_2} (b-\cos\theta)^{1/2} d\theta = \frac{2\sqrt{2}}{\sqrt{m}}\big(E(m,\alpha_1) - E(m,\alpha_2)\big), \quad (18)$$

$$\int_{\phi_1}^{\phi_2} (b+\cos\theta)^{1/2} d\theta = \frac{2\sqrt{2}}{\sqrt{m}}\big(E(m,\gamma_2) - E(m,\gamma_1)\big), \quad (19)$$

$$\int_{\phi_1}^{\phi_2} \frac{\cos\theta\, d\theta}{(b-\cos\theta)^{1/2}} = \sqrt{2m}\left(\frac{2-m}{m}\big(F(m,\alpha_2) - F(m,\alpha_1)\big)\right) - \frac{2\sqrt{2}}{\sqrt{m}}\big(E(m,\alpha_2) - E(m,\alpha_1)\big) \quad (20)$$

$$\int_{\phi_1}^{\phi_2} \frac{\cos\theta\, d\theta}{(b+\cos\theta)^{1/2}} = \sqrt{2m}\left(\frac{m-2}{m}\big(F(m,\gamma_2) - F(m,\gamma_1)\big)\right) + \frac{2\sqrt{2}}{\sqrt{m}}\big(E(m,\gamma_2) - E(m,\gamma_1)\big), \quad (21)$$

$$\int_{\phi_1}^{\phi_2} \frac{d\theta}{(b-\cos\theta)^{3/2}} = \sqrt{2m}\left(\frac{m}{2-2m}\right)\big(E(m,\omega_2) - E(m,\omega_1)\big), \quad (22)$$

$$\int_{\phi_1}^{\phi_2} \frac{\cos\theta\, d\theta}{(b-\cos\theta)^{3/2}} = \frac{(2-m)\sqrt{2m}}{2-2m}\big(E(m,\omega_2) - E(m,\omega_1)\big) - \sqrt{2m}\big(F(m,\omega_2) - F(m,\omega_1)\big), \quad (23)$$

where

$$\alpha_i = \left(\frac{\pi - \phi_i}{2}\right), \quad (24)$$

$$\gamma_i = \frac{\phi_i}{2}, \quad (25)$$

$$\omega_i = \arcsin\left[\frac{\sin(\phi_i/2)}{\sqrt{1 - m\cos^2(\phi_i/2)}}\right], \quad (26)$$

$$m = \frac{2}{b+1}. \quad (27)$$

When $\phi_1 = 0$ and $\phi_2 = \pi$, the expressions (16) – (23) are simplified to the expressions found in [9]. This can be proven by using equations (24) – (27), (82) and (85).

Generally, identities (16) – (21) can be used to solve problems involving the calculation of electric, magnetic, or even gravitational potentials whose forms involve terms like $1/|\vec{r}_2 - \vec{r}_1|$. On the other hand, the (22) and (23) equations are often used to figure out electric or magnetic fields because they are related to the terms $1/|\vec{r}_2 - \vec{r}_1|^3$.

IV. SOME EXAMPLES OF HOW THE IDENTITIES CAN BE USED

This section provides three classic examples of electromagnetism, to which some identities from the preceding section can be applied.

A. *Electric potential of ring segment with constant linear charge density*

According to Griffiths [10], the electric potential of a continuous linear distribution of charge at a point P can be determined as

$$V = \frac{1}{4\pi\epsilon_0}\int_l \frac{\lambda dl}{|\vec{r}_2 - \vec{r}_1|}, \quad (28)$$

where λ is the linear charge density and dl is a small length element and $|\vec{r_2} - \vec{r_1}|$ is the distance from the small length element to the point where the potential is determined.

For this case, we want to find an expression of the electric potential in a point $P(\rho,\beta,z)$ generated by a segment of a circular ring of $R\theta_1$ length with a constant linear charge density λ (see Fig. 1). Using equation (28) we have

$$V = \frac{1}{4\pi\epsilon_0}\int_0^{\theta_1} \frac{\lambda R d\theta}{\sqrt{R^2 + \rho^2 + z^2 - 2R\rho\cos(\beta - \theta)}}. \quad (29)$$

By changing the variable $\phi = \beta - \theta$ and using the identity (12), the expression (29) can be rewritten as

$$V = \frac{\lambda}{4\pi\epsilon_0}\sqrt{\frac{Rm}{\rho}}\big(F(m,\alpha_2) - F(m,\alpha_1)\big), \quad (30)$$

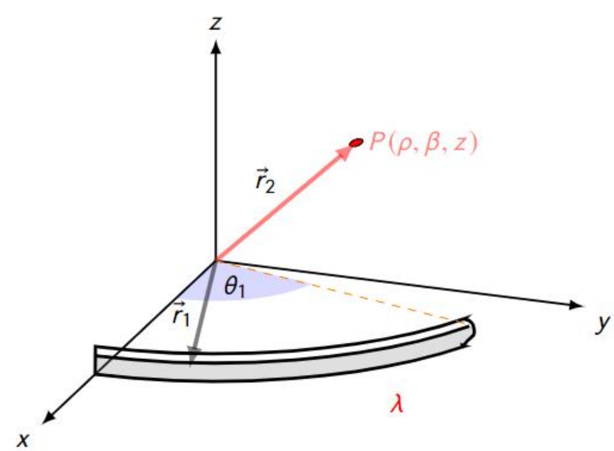

Fig. 1: Segment of a ring with constant charge density λ and length $R\theta_1$.

with

$$\alpha_1 = \frac{\pi - \beta}{2}, \tag{31}$$

$$\alpha_2 = \frac{\pi - \beta + \theta_1}{2}, \tag{32}$$

$$m = \frac{4R\rho}{(R+\rho)^2 + z^2}. \tag{33}$$

In the case that $\theta_1 = 2\pi$ (i.e., a circular ring) can be shown that the expression (30) is reduced to

$$V = \frac{\lambda}{2\pi\epsilon_0}\sqrt{\frac{Rm}{\rho}}F(m). \tag{34}$$

This corresponds to what is presented by Good [9], which makes sense because the electric potential of the ring is the same for all values of β when z and ρ remain constant.

Using equation (7), the expression (30) can be rewritten as

$$V = \frac{\lambda}{4\pi\epsilon_0}\sqrt{\frac{\mathrm{Rm}}{\rho}}\left(\frac{\theta_1}{2} + \frac{\mathrm{m}}{2}\int_{\alpha_1}^{\alpha_2}\sin^2\theta\,\mathrm{d}\,\theta + \frac{3\mathrm{m}^2}{8}\int_{\alpha_1}^{\alpha_2}\sin^4\theta\,\mathrm{d}\theta \right. \\ \left. \dots + \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}m^n\int_{\alpha_1}^{\alpha_2}\sin^{2n}\theta\,\mathrm{d}\theta\right). \tag{35}$$

Lastly, Fig. 2 shows, for different orders of approximation, how the electric potential changes as a function of z for the segment with $\theta_1 = 5\pi/6$ rad, $\beta = 3\pi/2$ rad y $\rho = 0.1$ m. It is evident from the figure that the difference between the first, second and third approximations order is negligible.

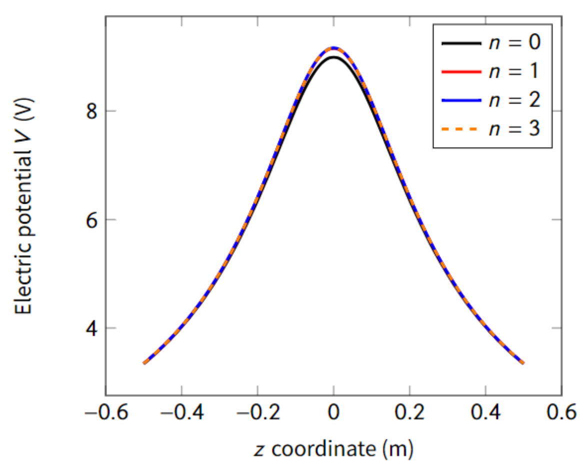

Fig. 2: Electric potential variation as a function of the z coordinate for different approximation orders for a ring segment. In this case, $\theta_1 = \frac{5}{6}\pi$, $\beta = \frac{6}{4}\pi$ y $\rho = 0.1$ m.

B. Magnetic field of ring segment carrying a constant current.

The magnetic field produced by an element that carries a current is determined as [11], [12]:

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int\frac{\mathrm{d}\vec{l}\times(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}. \tag{36}$$

In the present case, we focus on an element of a circular ring with a length $R\theta_1$. This element carries a constant current I in the counterclockwise direction (as depicted in Fig. 3. Consequently, the magnetic field created at a certain position $P(\rho, \beta, z)$ can be expressed as follows:

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_0^{\theta_1}\frac{zR(\cos\theta\hat{x} + \sin\theta\hat{y})\mathrm{d}\theta}{(R^2 + \rho^2 + z^2 - 2\rho R\cos(\beta - \theta))} \\ + \frac{\mu_0 I}{4\pi}\int_0^{\theta_1}\frac{(R^2 - \rho R\cos(\beta - \theta))\hat{z}\mathrm{d}\theta}{(R^2 + \rho^2 + z^2 - 2\rho R\cos(\beta - \theta))}. \tag{37}$$

Using the equations (22) and (23), the magnetic field components are expressed in terms of elliptic integrals.

$$B_x = \frac{\mu_0 Iz\cos\beta}{8\pi}\sqrt{\frac{m}{\rho^3 R}}(A - C) + \frac{\mu_0 Iz\sin\beta}{4\pi\rho}(D_2 - D_1), \tag{38}$$

$$B_y = \frac{\mu_0 Iz\sin\beta}{8\pi}\sqrt{\frac{m}{\rho^3 R}}(A - C) - \frac{\mu_0 Iz\cos\beta}{4\pi\rho}(D_2 - D_1), \tag{39}$$

$$B_z = \frac{\mu_0 I}{8\pi}\sqrt{\frac{m}{\rho^3 R}}(G - \rho\, A), \tag{40}$$

with

$$A = A(m) = \big(F(m,\omega_2) - F(m,\omega_1)\big), \tag{41}$$

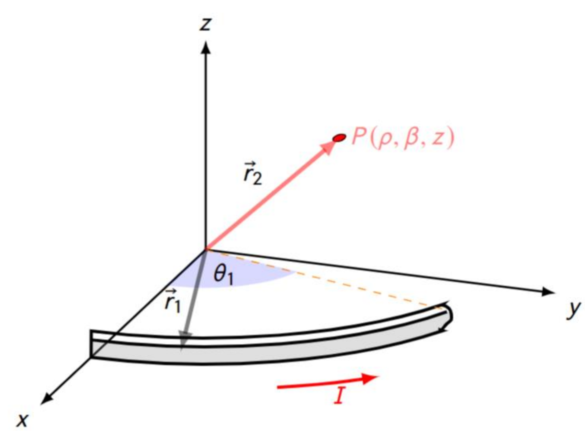

Fig. 3: Segment of a ring of $R\theta_1$ length and carrying a constant electric current I in the counterclockwise direction.

$$C = C(m) = \frac{2-m}{2-2m}\left(E(m,\omega_2) - E(m,\omega_1)\right), \quad (42)$$

$$D_k = D(\phi_k) = \frac{1}{\sqrt{R^2 + \rho^2 + z^2 - 2\rho R \cos\phi_k}}, \quad (43)$$

$$G = G(m) = \frac{2\rho - m(\rho + R)}{2-2m}\left(E(m,\omega_2) - E(m,\omega_1)\right), \quad (44)$$

$$\omega_k = \arcsin\left[\frac{\sin(\phi_k/2)}{\sqrt{1 - m\cos^2(\phi_k/2)}}\right], \quad (45)$$

$$m = \frac{4R\rho}{(R+\rho)^2 + z^2}, \quad (46)$$

$$\phi_1 = \beta, \quad (47)$$

$$\phi_2 = \beta - \theta_1. \quad (48)$$

The B_z component can be expressed using equations (7) and (11) as:

$$B_z = K[S\eta - I_1\rho + (J_1 + L_1)m + \cdots + (J_n - L_n)m^n], \quad (49)$$

where

$$J_n = S\left(\eta - \frac{1}{2}I_1 - \cdots - \frac{1\cdot 3\cdot 5\cdots(2n-3)}{2\cdot 4\cdot 6\cdots 2n}I_n\right), \quad (50)$$

$$L_n = \left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right)\rho I_{n+1}, \quad (51)$$

$$I_n = \int_{\omega_1}^{\omega_2} \sin^{2n}\theta \, d\theta, \quad (52)$$

$$\eta = \omega_2 - \omega_1, \quad (53)$$

$$K = \frac{\mu_0 I}{8\pi}\sqrt{\frac{m^3}{\rho R}}, \quad (54)$$

$$S = \left(\frac{\rho - R}{2}\right). \quad (55)$$

Fig. 4 shows the change in the axial component of the magnetic field produced by a semicircular (this means that $\theta_1 = \pi$ is used in equation 49) wire carrying a constant current I in the counterclockwise direction as a function of the z coordinate. The figure displays the approximations of order $n = 0, 1$ and 5, along with the outcome of a numerical simulation conducted using the Comsol Multiphysics software. In all cases, the ring is considered to have the following physical properties: $R = 0.1$ m, $I = 1$ A, $\rho = 0.001$ m, and $\beta = \pi$.

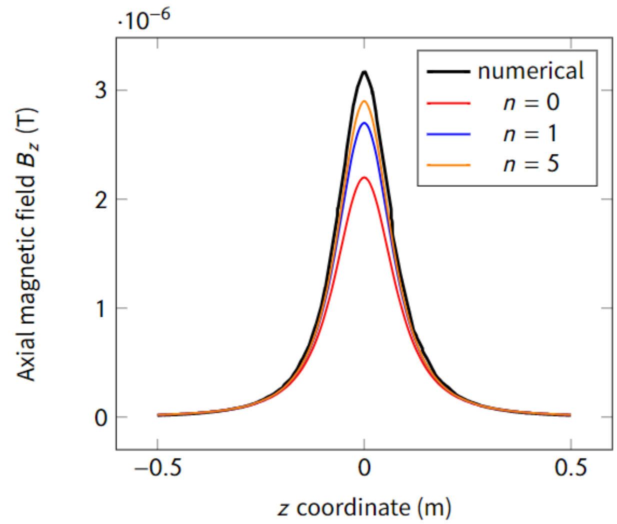

Fig. 4:Variation of the axial magnetic field as a function of z for different approximation orders for a ring segment with a constant I current in the counterclockwise direction. In this case, $\theta_1 = \beta = \pi$ and $\rho = 0.001$ m.

Finally, in Fig. 4, it can be observed that as the order of approximation increases, the value of the axial component of the field tends to approach the value obtained in the simulation.

C. *Magnetic Field of a cylindrical permanent magnet.*

According to Reitz [12], Van Tai Nguyen and Tien-Fu Lu [13] and Fontana [14] the magnetic field produced by permanent magnets can be determined using the Couloumbian model, frequently referred to as the method of magnetic charge, that is

$$\vec{B} = -\mu_0 \nabla \varphi_M, \quad (56)$$

with

$$\varphi_M = \frac{1}{4\pi}\int_{V_1}\frac{\rho_M dV_1}{|\vec{r}_2 - \vec{r}_1|} + \frac{1}{4\pi}\int_{A_1}\frac{\sigma_M dA_1}{|\vec{r}_2 - \vec{r}_1|}, \quad (57)$$

where ρ_M and σ_M are the magnetic pole density and surface density of the magnetic pole intensity, respectively, which are defined as

$$\rho_M = -\nabla \cdot \vec{M}, \quad (58)$$

$$\sigma_M = \vec{M} \cdot \vec{n}. \quad (59)$$

In this case, a segment of an annular permanent magnet with constant magnetization $\vec{M} = M\hat{z}$ is considered (Fig. 5). The magnet has an internal radius r_1, external radius r_2 and height $h_2 - h_1$.

As the permanent magnet has a constant magnetization, the magnetic field in $P(\rho, \beta, z)$ is given by the expression

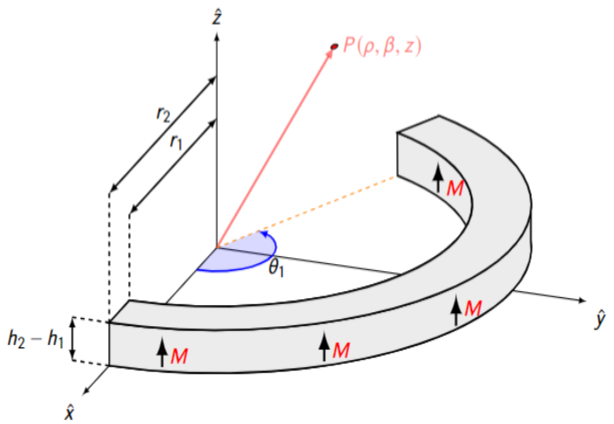

Fig. 5: Segment of an annular permanent magnet with axial magnetization.

$$\vec{B} = \frac{\mu_0 M}{4\pi} \int_{r_a}^{r_b} \int_0^{\theta_1} [\overrightarrow{\mathcal{B}_b} - \overrightarrow{\mathcal{B}_a}] r \, d\theta dr\,, \tag{60}$$

where

$$\overrightarrow{\mathcal{B}_j} = \frac{(\rho\cos\beta - r\cos\theta)\hat{x} + (\rho\sin\beta - r\sin\theta)\hat{y}}{\left[r^2 + \rho^2 + (z - h_j)^2 - 2r\rho\cos(\beta - \theta)\right]^{3/2}} + \frac{(z - h_j)\hat{z}}{\left[r^2 + \rho^2 + (z - h_j)^2 - 2r\rho\cos(\beta - \theta)\right]^{3/2}}. \tag{61}$$

Using the identities (22) y (23) the components of $\vec{B}$ are

$$B_x = \frac{\mu_0 M \cos\beta}{8\pi} \int_{r_1}^{r_2} \frac{r dr}{(\rho r)^{3/2}} \left(A(m_2) - A(m_1) - C(m_2) + C(m_1)\right) - \frac{\mu_0 M \sin\beta}{4\pi\rho}[D_{222} - D_{122} - D_{221} + D_{121}] + \frac{\mu_0 M \sin\beta}{4\pi\rho}[D_{212} - D_{112} - D_{211} + D_{111}], \tag{62}$$

$$B_y = \frac{\mu_0 M \sin\beta}{8\pi} \int_{r_1}^{r_2} \frac{r dr}{(\rho r)^{3/2}} \left(A(m_2) - A(m_1) - C(m_2) + C(m_1)\right) + \frac{\mu_0 M \cos\beta}{4\pi\rho}[D_{222} - D_{122} - D_{221} + D_{121}] - \frac{\mu_0 M \cos\beta}{4\pi\rho}[D_{212} - D_{112} - D_{211} + D_{111}], \tag{63}$$

$$B_z = \frac{\mu_0 M}{16\pi} \int_{r_1}^{r_2} \frac{r dr}{(\rho r)^{3/2}} \left((h_2 - z)G(m_2) - (h_1 - z)G(m_1)\right), \tag{64}$$

with

$$A(m_j) = \frac{2r - (\rho + r)m_j}{(2 - 2m_j)} \sqrt{m_j} \left(E(m_j, \omega_{j2}) - E(m_j, \omega_{j1})\right), \tag{65}$$

$$C(m_j) = r\sqrt{m_j}\left(F(m_j, \omega_{j2}) - F(m_j, \omega_{j1})\right), \tag{66}$$

$$D_{ijk} = \sqrt{r_i^2 + \rho^2 + (z - h_j)^2 - 2\rho r_i \cos\phi_k} + \rho\cos\phi_k \tanh^{-1}\left(\frac{2r_i - 2\rho\cos\phi_k}{2\sqrt{r_i^2 + \rho^2 + (z - h_j)^2 - 2\rho r_i\cos\phi_k}}\right) \tag{67}$$

$$G(m_j) = \frac{m_j^{3/2}}{1 - m_j}\left(E(m_j, \omega_{j2}) - E(m_j, \omega_{j1})\right), \tag{68}$$

$$\omega_{jk} = \arcsin\left[\frac{\sin(\phi_k/2)}{\sqrt{1 - m_j\cos^2(\phi_k/2)}}\right], \tag{69}$$

$$m_j = \frac{4r\rho}{(r + \rho)^2 + (z - h_j)^2}, \tag{70}$$

$$\phi_1 = \beta, \tag{71}$$

$$\phi_2 = \beta - \theta_1, \tag{72}$$

where the i, j, and k indices of the D_{ijk} function correspond to the values of r_i (r_1, r_2), h_j (h_1, h_2) y ϕ_k (ϕ_1, ϕ_2).

Using equation (11), the expression (64) can be rewritten as

$$B_z = \kappa(h_2 - z)\int_{r_1}^{r_2} r d r \left(\frac{m_2}{\rho r}\right)^{3/2} [\eta_j + J_{21}m_2 + \cdots + J_{2n}m_2^n] - \kappa(h_1 - z)\int_{r_1}^{r_2} r d r \left(\frac{m_1}{\rho r}\right)^{3/2} [\eta_j + J_{11}m_1 + \cdots + J_{1n}m_1^n], \tag{73}$$

with

$$J_{jn} = \left(\eta_j - \frac{1}{2}I_1 - \cdots - \frac{1\cdot 3\cdot 5\cdots(2n-3)}{2\cdot 4\cdot 6\cdots 2n}I_{jn}\right), \tag{74}$$

$$I_{jn} = \int_{\omega_{j1}}^{\omega_{j2}} \sin^{2n}\theta d\theta, \tag{75}$$

$$\eta_j = \omega_{j2} - \omega_{j1}, \tag{76}$$

$$\kappa = \frac{\mu_0 M}{16\pi}, \tag{77}$$

where n corresponds to natural numbers and j can only take on the values 1 or 2, depending on the case of m_j.

Finally, in Fig. 7 and Fig. 8 the approximations $n = 0, 1$ and 2 are shown in addition to the result of a numerical simulation carried out with the Comsol Multiphysics software for the axial component of the magnetic field of a permanent ring magnet (this means that $\theta_1 = 2\pi$ is used in equation (64)) with constant axis magnetization $\vec{M} = M\vec{k}$ (see Fig. 6 in the function of z. In both figures, the magnet is considered to have the following physical properties: $r_1 = 26.6$ mm, $r_2 = 34.8$ mm, $h_1 = 0$, $h_2 = 20$ mm, and $M = 10$ kA/m. In Fig. 7, B_z was determined at $\rho = 0.01$ m and $\beta = 0$, whereas for Fig. 8, B_z has been determined at $\rho = 0.001$ m and $\beta = 0$. When comparing the two figures, lower ρ values correspond more closely with the numerical simulation value.

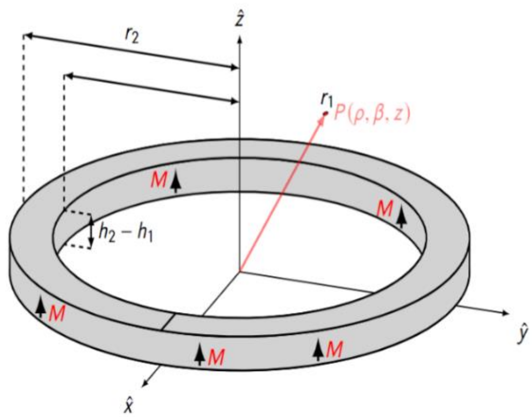

Fig. 6: Annular Permanent magnet with axial magnetization.

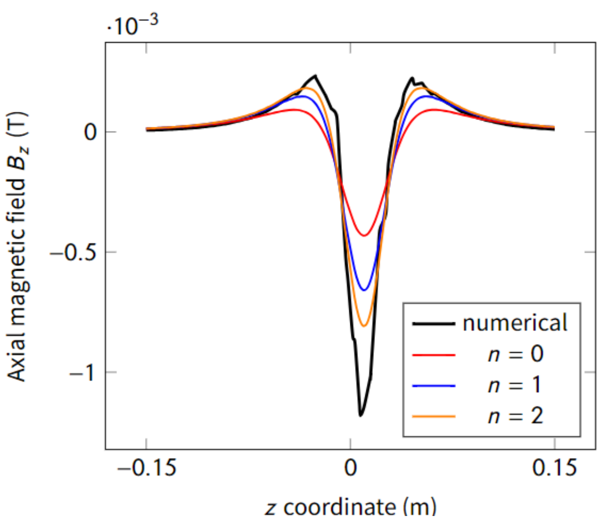

Fig. 7: Axial magnetic field variation as a function of z for different approximation commands for an axially magnetized ring. In this case: $\theta_1 = 2\pi$, $\beta = 0$ and $\rho = 0.01$ m.

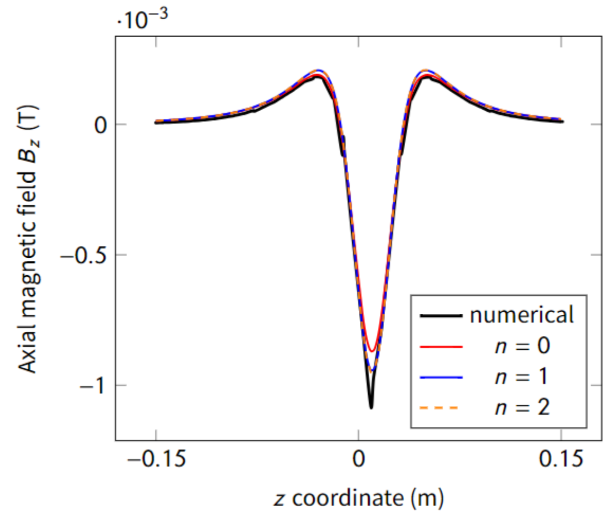

Fig. 8:Axial magnetic field variation as a function of z for different approximation commands for an axially magnetized ring. In this case: $\theta_1 = 2\pi$, $\beta = 0$ and $\rho = 0.001$ m.

V. Conclusion

Thanks to this work, it was possible to provide a list of identities involving incomplete elliptic integrals, which serve as support to engineering and physics students in solving problems typical of electromagnetism. Specifically, three sine function identities and seven cosine function identities relating to the incomplete elliptic integrals of the first kind and the second kind were given. Consequently, some of the identities could be applied to the solution of classic electromagnetism problems. Finally, this publication is expected to motivate students to use and learn more about incomplete elliptic integrals.

Appendix: Elliptic Integral Fundamental Relations.

This section presents some identities that can be very useful when working with incomplete elliptic integrals, which can be found in Byrd [4]

$$E(m, -\varphi) = -E(m, \varphi), \quad (78)$$

$$F(m, -\varphi) = -F(m, \varphi), \quad (79)$$

$$E(m, n\pi \pm \varphi) = 2nE(m) \pm E(m, \varphi), \quad (80)$$

$$F(m, n\pi \pm \varphi) = 2nF(m) \pm F(m, \varphi), \quad (81)$$

$$E(m, \alpha) \pm E(m, \beta) = E(m, \varphi) \pm m \sin\alpha \sin\beta \sin\varphi\,, \quad (82)$$

$$F(m, \alpha) \pm F(m, \beta) = F(m, \varphi), \quad (83)$$

where

$$\varphi = 2\arctan\left[\frac{\sin\alpha\sqrt{1 - m\sin^2\beta} \pm \sin\beta\sqrt{1 - m\sin^2\alpha}}{\cos\alpha + \cos\beta}\right] \quad (84)$$

or

$$\varphi = \arccos\left[\frac{\cos\alpha\cos\beta}{1 - m\sin^2\alpha\sin^2\beta} \mp \frac{\sin\alpha\sin\beta\sqrt{(1 - m\sin^2\alpha)(1 - m\sin^2\beta)}}{1 - m\sin^2\alpha\sin^2\beta}\right] \quad (85)$$

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