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Mathematical Models for Planning Combat Strategy and Decision Making

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***Abstract*– Math applications are practically in all the fields of sciences unlimitedly. From the use of cryptography, the science of securing and transmitting data with the use of matrix algorithms till the use of variational calculus for the intersection of ballistic missiles, mathematics is present in every aspect of military sciences. The study and analysis of military from a mathematical perspective has led to the construction of the mathematical models of combat. These models describe the evolution and predict the outcome of a military combat with the use of robust mathematical criteria. This will help to derive the correct strategy to use as well as the correct decision making to defeat the opponent and avoid disastrous mistakes.**

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Keywords—equations, firepower, combat, linear, square

I. Introduction

In this article the author describes and elucidates through real applications two of the most known mathematical models of combat; Lanchester's square law of direct fire and Lanchester's linear law of undirect fire. The article analyzes and shows the importance and influence of both models of combat on military science. Using the square laws model the 3:1 advantage ratio is derived to show how by outnumbering the opponent three times can lead to its neutralization and defeat. To illustrate the applications of the models two real of battles are analyzed and modeled using the square and the linear law, the Iwo Jima battle and the Alamo battle.
Finally the author demonstrates how both laws and their combination can extensively guide to plan the best strategy and decision making when fighting rogue states and guerrilla groups whose main tactics rely on asymmetric warfare.

II. OVERWIEW ABOUT LANCHESTER'S MATHEMATICAL MODELS

A. Lanchester's Mathematical Models of Combat and the Laws of Direct and Undirected Fire

In the middle of World War I (1916), the outstanding British engineer and polymath Frederick Lanchester, a pioneer of the aeronautic and automobile industries developed a mathematical model to predict the evolution and outcome of a military combat. The model i constituted by a system of differential equations that relate the variation of the number of troops with regard to time as a function of the number of troops and their firepower [3]. An important result of Lanchester equations are the square law model of direct fire and the linear law model of indirect fire.

Due to their nature, the mathematical models of combat are classified into discrete and stochastic models. Both types of models are based on systems of discrete and stochastic differential equations respectively as well as on the theory of linearized stability of dynamic systems [1]. In practice the most used models are the Lanchester laws of direct and undirect fire, Kolmogorov discrete model [2] and Markov stochastic model. Among them, the Lanchester model stands out as the most direct and practical one. Kolmogorov and Markov mathematical models require much more information and operations for their executions.

A.1 *The Square Law of Direct Fire*

Direct fire involves the situation when two armies fire each other knowing their exact positions. To derive the square law of direct fire let's consider two armies in combat under direct fire; army one and army two, with a number of troops equals to $x_1(t)$ and $x_2(t)$ in the instant of time t respectively. Therefore, the variation of the number of troops of each army with respect to time is directly proportional to the opponent's number of troops and to its firepower [4]. Hence the variation of troop numbers of both armies is given by following the system of differential equations:

$$\begin{cases} \dot{x}_1(t) = -a_2 x_2 & (1) \\ \dot{x}_2(t) = -a_1 x_1 & (2) \end{cases}$$

where $\dot{x}_1(t)$ and $\dot{x}_2(t)$ *are the derivatives of the functions, $x_1(t)$* and *$x_2(t)$*, a_1 and a_2 their firepower (number of inflicted casualties per unit of time by a soldier of army one and army two respectively). If it is known that at the beginning of the battle, when t = 0, both armies have an initial number of soldiers equal *to x_{10}* and x_{20}, then after solving the system of differential equations we get:

$$x_1 = \sqrt{\frac{a_2(x_2{}^2 - x_{20}{}^2) + a_1 x_{10}{}^2}{a_1}} \quad \text{and} \quad x_2 = \sqrt{\frac{a_1(x_1{}^2 - x_{10}{}^2) + a_2 x_{20}{}^2}{a_2}}.$$

From the solution of the system, we obtain the condition:

$$a_1 x_1{}^2 - a_2 x_2{}^2 = a_1 x_{10}{}^2 - a_2 x_{20}{}^2 = C,$$

where C is a constant unchangeable through time. Hence army one defeats army two under direct fire when $C > 0$, that is under the condition:

$$a_1 x_{10}{}^2 > a_2 x_{20}{}^2.$$

Similarly, army two defeats army one under direct fire when $C < 0$, that is when:

$$a_2 x_{20}{}^2 > a_1 x_{10}{}^2.$$

Finally, if $C = 0$, then:

$$a_1 x_{10}{}^2 = a_2 x_{20}{}^2$$

In this case both armies fight to a mutual stalemate.

The 3:1 Force Ratio for the Victory Under Direct Fire

From the square law of direct fire, it follows that army two is defeated under the condition:

$$a_2 x_{20}{}^2 > a_1 x_{10}{}^2.$$

No, assume that army one firepower is double army two firepower and the initial number of troops of army two is twice larger than the initial number of troops of army one. If that is the case, we shall have:

$$a_1 = 2a_2\,,\ \ x_{20} = 2x_{10} \Rightarrow 2a_2 x_{10}{}^2 < a_2(2x_{10})^2 = 4a_2 x_{10}{}^2.$$

Even when $a_1 = 3a_2,\ x_{20} = 2x_{10} \Rightarrow 3a_2 x_{10}{}^2 < 4a_2 x_{10}{}^2.$

Hence having a firepower three times greater will not defeat an army twice bigger. Only increasing the firepower four times with respect to the opponent can neutralize it even if it is twice bigger.

Finally, if $x_{20} = 3x_{10}$, then:

$$\mathbf{2}a_2 x_{10}{}^2 < a_2(3x_{10})^2 = 9a_2 x_{10}{}^2$$
$$\mathbf{3}a_2 x_{10}{}^2 < a_2(3x_{10})^2 = 9a_2 x_{10}{}^2$$
.
.
.
$$\mathbf{8}a_2 x_{10}{}^2 < a_2(3x_{10})^2 = 9a_2 x_{10}{}^2$$

To illustrate the evolution of combat under direct fire with a variation of firepower and an increase in the number of troops, Three simulations were carried out and the evolution and outcome of combat are shown in the graphs. The following three simulations illustrate the evolution of combat under direct fire with an increase of fire power in army one in the first and second simulations and an increase of the number of troops of army two in the third simulation.

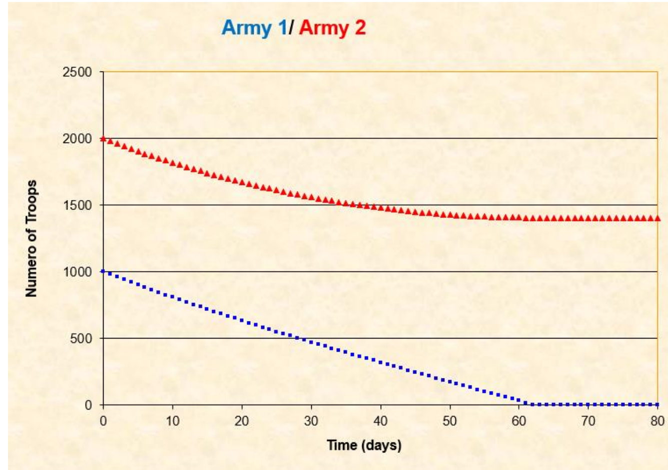

Fig. 1 Evolution of a combat with $a_1 = 2a_2$, $x_{20} = 2x_{10}$

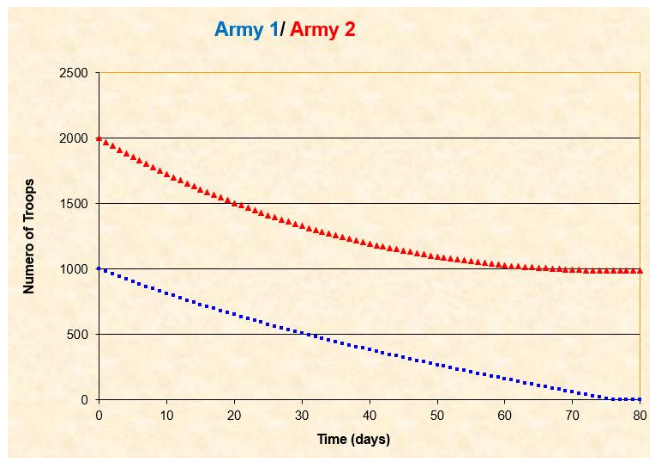

Fig.2 Evolution of a combat with $a_1 = 3a_2$, $x_{20} = 2x_{10}$

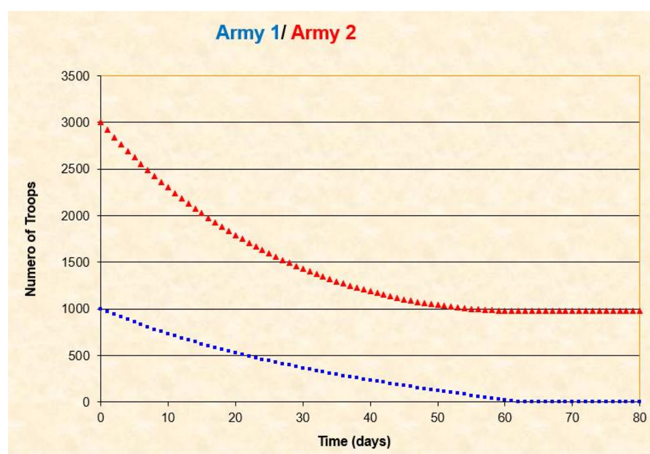

Fig.3 Evolution of a combat with $a_1 = 8a_2$, $x_{20} = 3x_{10}$

The simulations show that in a fight against an opponent army eight times more efficient under direct fire, to have at least three times more troops than the opponent army can neutralize it and eventually lead to its defeat.

An evidence of this was observed during the Falklands war that ended up with the surround of the Argentinian army by British troops in 1981. In this war the numbers of Argentinian and British troops were 23428 and 25 948 respectively. Being outnumbered and outgunned by the British by a fire power two and half times greater, Argentina didn't have a have a chance to win the war, unless it would have put at least 78 000 troops on the ground to take advantage of the square law of direct fire. As a result of this strategical mistake Argentina lost the war after surrendering to Britain.

A.2 *The Linear Law of Indirect Fire*

Indirect fire involves the case when two armies fire each other without knowing their exact positions, but just the area where their troops are located [6].

To derive the linear law of indirect fire let's consider two armies in combat under indirect fire; army one and army two, with a number of troops equals to $x_1(t)$ and $x_2(t)$ in the instant of time t respectively. Therefore if $x_1(t)$ fires indirectly to $x_2(t)$, the variation of the number of troops of army two with respect to time is directly proportional to $x_1(t)$ and $x_2(t)$ as well as to army's one firepower. Hence the variation of troop numbers of both armies is given by the system of differential equations:

$$\begin{cases} \dot{x_1}(t) = -a_2 x_1 x_2 & (3) \\ \dot{x_2}(t) = -a_1 x_1 x_2 & (4) \end{cases}$$

Solving the system of differential equations, we obtain:

$$x_1 = \frac{a_2}{a_1}(x_2 - x_{20}) + x_{10} \quad \text{and} \quad x_2 = \frac{a_1}{a_2}(x_1 - x_{10}) + x_{20}.$$

$$a_1 x_1 - a_2 x_2 = a_1 x_{10} - a_2 x_{20} = C,$$

where C is a constant unchangeable through time. Hence army one defeats army two under in direct fire when $C > 0$, that is under the condition:

$$a_1 x_{10} > a_2 x_{20}.$$

Similarly, army two defeats army one under direct fire when $C < 0$, that is:

$$a_2 x_{20} > a_1 x_{10}.$$

Finally, if C = 0, then:

$$a_1 x_{10} = a_2 x_{20}.$$

In this case both armies fight to a mutual stalemate.

Therefore, in a combat under indirect fire, a numerical advantage and a superior firepower are crucial to defeat the opponent [7]. Nevertheless, an optimal combination of both produces better outcomes.

A practical example of an application of the linear law of indirect fire was in the strategy used by the coalition lead by The United States in the Gulf war. Having deployed more than 500 0000 troops to the region to liberate Kuwait, the USA used indirect fire in the first stages of operation Desert Storm to breach the enemies' defenses and thus to prepare the ground for a land invasion. Over 14,000 rounds were fired during these missions destroying 22 artillery battalions, including the destruction of approximately 396 artillery pieces. Indirect fire proves to be critical when planning the strategy for a land occupation, where the first goal is to breach and decimate the enemies' defense with the use of indirect fire.

Fig. 4 U.S. artillery firing during the Gulf war

A.3. *Asymmetric Warfare and the Fight against Occupational States*

The fight against guerrilla groups has an asymmetric nature due to the tactics applied by the insurgents. An army will be under direct fire when is suppressively ambushed by the rebels, while the rebels will become under indirect fire when the army returns fire to the rebels' unknown position. As result his type of fight can be modeled by a combination of the square law of indirect fire and the linear law of direct fire. Let $x(t)$ and $y(t)$ be the number of troops and rebels in the instant of time t respectively and let α and β be their firepower. Then we'll have a system of two equations, where the first one describes a direct fire scenario of combat while and the second one an indirect fire scenario.

$$\begin{cases} \frac{dx}{dt} = -\beta y & (4.4) \\ \frac{dy}{dt} = -\alpha x y & (4.8) \end{cases}$$

Solving the system of differential equations:

$$x = \sqrt{\frac{2\beta}{\alpha}(y - y_o) + {x_0}^2}$$

Hence the insurgents' defeat occurs when:

$$\alpha {x_0}^2 > 2\beta y_0$$

That is,

$$y_o < \frac{\alpha {x_0}^2}{2\beta}$$

To provide this inequality, β should be the smallest possible, which is beyond the army's control. Therefore, the only way to hold the inequality will be by increasing the numerator of the fraction. Hence if increasing x_0 by √2 times we'll get:

$$\frac{\alpha(\sqrt{2}x_0)^2}{2\beta} = \frac{\alpha {x_0}^2}{\beta}$$

Then

$$y_o < \frac{\alpha {x_0}^2}{2\beta},$$

Provided that x_0 is increased at least by √2 times, i,e.,

$$>> \sqrt{2}\, x_0$$

Lanchester's law can be very useful to design the strategy for fighting against a rogue state aggression or invasion. In this case the square and the linear laws will be used according to the stages and scenarios of combat. Also, a transformation of the square law into the linear law can be possible thanks to the smart use of technology. A real-life example of this has been observed in the Ukraine - Russia war, when the Russian troops advanced in the offensive to capture and take control of Ukrainian territories. This happened during the battle of the Siverskiy Donetz, when Ukrainian artillery used indirect fire to hit precisely an entirely battalion. Ukraine was able to use indirect fire weapons to hit the target directly as in the case of direct fire. The reason behind this was the use of drones to spot the precise position of the Russian forces, and send this information to the artillery detachment to calibre their weapons to hit then precisely. As it can be seen, the use of drone's technology allows to take advantage of indirect fire weapons for direct fire. The transformation from indirect fire to direct fire with the use of drones goes as follows:

$$\dot{x_2} = -\,a_1 x_1 x_2 \quad \Longrightarrow \quad \dot{x_2} = -\,a_1 x_1$$

III. DEVELOPMENT OF PROPOSAL

To illustrate the use an application of the mathematical models of combat, two battle models will be built based on systems of differential equations; the model of the battle of Iwo Jima battle and the model of the Alamo battle. Also, a combination of the square law and linear law of fire is applied to derive the mathematical model of an asymmetric combat and thus to obtain an important result to be considered when fighting irregular armies. The systems of differential equations for the first two models will be solved taking into consideration the initial conditions of the battles. The obtained results give critical information about the firepower force ratio of the fighting armies, and shows how it changes depending on the nature of the stage of combat.

A. Mathematical Model of the Battle of Iwo Jima

Fig. 5 U.S. soldiers rising the American flag in Iwo Jima

The Iwo Jima battle is considered as one of the toughest battles of World War Two fought in the Iwo Jima Island between the American and Japanese forces [8]. To build up the mathematical model of the battle, the reinforcement of American troops is taken into account for the calculations.
Let $x_1(t)$ and $x_2(t)$ be the number of American and Japanese troops at the instant of time t and let x_{10} and x_{20} be their initial numbers of troops. Then, the system of differential equations that describe the battle under direct fire according to the nature of this battle is:

$$\begin{cases} \dot{x}_1(t) = -\beta x_2 + R_1(t) & (5) \\ \dot{x}_2(t) = -\,\alpha\, \dot{x}_1(t) & (6) \end{cases}$$

Where α is the American firepower, β is the Japanese firepower and R_1 (t) the number of American reinforcements arriving daily as shown in figure 5 below.

Reinforcement rate of American soldiers per day $R_1(t)$	Time given in days t
5400	$0 < t \leq 1$
0	$1 < t \leq 2$
6000	$2 < t \leq 3$
0	$3 < t \leq 4$
0	$4 < t \leq 5$
13000	$5 < t \leq 6$
0	$6 < t \leq 36$

Fig. 5 Daily reinforcement rate of American troops

It is known that at the beginning of the battle $x_{10} = 0$ and $x_{20} = 22500$.

Integrating (6):

$$x_2(36) - x_2(0) = -\alpha \int_0^{36} x_1(t)dt = -\alpha \sum_{i=1}^{36} x_i(t) \qquad (7)$$

$$\Rightarrow \alpha = \frac{x_2(0) - x_2(36)}{\sum_{i=1}^{36} x_i(t)} = 0.0106$$

Integrating (5) from $t = 0$ to $t = 36$:

$$\dot{x}_1(t) = -\beta x_2 + R_1(t)$$

$$x_1(36) - x_1(0) = -\beta \int_0^{36} x_2(t)dt + \int_0^{36} R_1(t)dt$$

$$\Rightarrow \beta = \frac{\int_0^{36} R_1(t)dt - x_1(36) + x_1(0)}{\int_0^{36} x_2(t)dt}$$

$$= \frac{73000 - x_1(36) + x_1(0)}{\int_0^{36} x_2(t)dt} \qquad (8)$$

From (7):

$$x_2(j) = x_2(0) - \alpha \sum_{i=1}^{j} x_i(t)$$

Using x_2 in (8) to compute the integral $\int_0^{36} x_2(t)dt$

$$\Rightarrow \beta = \frac{\int_0^{36} R_1(t)dt - x_1(36) + x_1(0)}{\int_0^{36} x_2(t)dt} = 0.00543$$

B. Mathematical Model of the Battle of the Alamo

Fig. 6 The Alamo fortress

Due to its combat characteristic the battle of the Alamo (February 23 – March 6, 1836) between the Mexican army and Texan defenders can be modeled by the square law of direct fire and the linear law of indirect fire. The model of the battle has two phases. The first phase depicts an asymmetric combat scenario while the second one a symmetric combat scenario. Let $x_1(t)$ and $x_2(t)$ be the number of Texan and Mexican troops at the instant of time t. Let x_{10} and x_{20} be their initial numbers of troops and let a_1 and a_2 be their firepower. For both phases of the battle, we have the following data:

$x_{10} = 188$, $x_{20} = 2400$: number of soldiers at beginning of the battle.
$x_{11} = 100$, $x_{21} = 1800$: number of soldiers at the end of the first phase of the battle.
$x_{12} = 100$, $x_{21} = 1800$: number of soldiers at the beginning of the second phase of the battle.
$x_1 = 0$ and $x_2 = 1750$: number of soldiers at the end of the second phase of the battle.

B.1 First Phase of the Battle

In the first phase of the battle the Texan defenders were protected inside the fort, while the Mexican army was outside in the open. The Texans use direct fire against the Mexican, while the Mexican returned undirected fire. As a result, the Mexicans suffered severe losses and at the end of the first phase only 1800 Mexican and 100 Texan were still fighting. Due to the nature of combat the first phase of the battle is described by the linear law of undirected fire and the square law of directed fire. Thus,

$$\begin{cases} \dot{x}_1(t) = -a_2 x_1(t) x_2(t) & (9) \\ \dot{x}_2(t) = -a_1 x_1(t) & (10) \end{cases}$$

After separating variables and integrating we get,

After separating variables and integrating we get,

$$\frac{a_1}{a_2} = \frac{1}{2}\left(\frac{{x_2}^2 - {x_{20}}^2}{x_1 - x_{10}}\right)$$

Replacing, in the expression above we obtain:

$$\frac{a_1}{a_2} = \frac{1}{2}\left(\frac{1800^2 - 2400^2}{100 - 188}\right)$$

$$\frac{a_1}{a_2} = 14318$$

Hence during the first phase of the battle the effectiveness of the Texan men was extremely superior to the Mexican men due to the advantage of their location and to their fierce resistance.

B.2 Second Phase of the Battle

In the second phase of the battle the Mexican army was able to breach the fort defenses and the combat was fought under direct fire. Therefore, the second phase of the battle is described by the square law of directed fire. Hence the firepower of the Texan defenders and the Mexican army will differ from the one in the first phase. Let b_1 *and* b_2 be their firepower respectively. Then,

$$\begin{cases} \dot{x}_1(t) = -b_2 x_2(t) & (11) \\ \dot{x}_2(t) = -b_1 x_1(t) & (12) \end{cases}$$

Separating variables and integrating:

$$b_1 {x_1}^2 - b_2 {x_2}^2 = b_1 {x_{12}}^2 - b_2 {x_{21}}^2$$

Replacing the values $x_1 = 0$, $x_2 = 1750$, $x_{12} = 100$, $x_{21} = 1800$:

$$-b_2(1750^2) = b_1(100^2) - b_2(1800^2)$$
$$\Longrightarrow \quad \frac{b_1}{b_2} = 18$$

Hence the effectiveness of the Texan defenders during the second phase of the battle declines considerably, yet it is still superior to the effectiveness of the Mexican army. This is because the Texan resisted ferociously without giving up. Nevertheless, under directed fire, the numerical superiority of the Mexican army was the key factor for the victory.

C. Construction of the Mathematical Model of Asymmetric Combat

To build a mathematical model of asymmetric combat, we will use the scenario of a combat between a conventional army fighting and an insurgent army that uses non-conventional fighting techniques such as ambushes and suppressive attacks [9]. In this case the army will be under direct fire while the insurgents under indirect fire.

Due to the nature of combat, the system of differential equations that describes the fight is made up of two equations; the first one is a direct fire equation, while the second one an indirect fire equation. Let x(t) and y(t) be the number of troops of the conventional army and the number of insurgents respectively, and let α and β be their firepower.

Then the system of differential equations that describe the mathematical model of the combat is given by:

$$\begin{cases} \frac{dx}{dt} = -\beta y & (13) \\ \frac{dy}{dt} = -\alpha x y & (14) \end{cases}$$

Dividing (13) by (14):

$$\frac{dx}{dy} = \frac{\beta}{\alpha x} \quad \Rightarrow \alpha x dx = \beta dy$$

$$\Longrightarrow x = \sqrt{\frac{2\beta}{\alpha}(y - y_o) + {x_0}^2}$$

$$\alpha x^2 - 2\,\beta y = \alpha {x_0}^2 - 2\,\beta y_o = C,$$

Where C is a constant unchangeable through. Therefore, the army defeats the insurgents when $C > 0$, that is under the condition:

$$\alpha {x_0}^2 > 2\,\beta y_o.$$

$$\Rightarrow y_o < \frac{\alpha {x_0}^2}{2\beta}$$

IV. RESULTS

From the mathematical model of the Iwo Jima battle we were able to calculate the firepower of the American and Japanese troops by operating the system of equations that describe the model. Thus, as a result α = 0.016 and β = 0.00543.

Hence:

$$\frac{\beta}{\alpha} = \frac{0.0543}{0.0106} = 5.1$$

$$\Longrightarrow \beta = 5{,}1\alpha$$

Therefore, we can conclude that the firepower of the Japanese troops was five times greater that the firepower of the American troops.

The mathematical model of the battle of the Alamo shows how much the firepower of an army can change when fighting under direct and indirect fire. It was possible to see how in the first phase of the battle the firepower of the Texan defenders easily overcomes the Mexicans' one, thanks to their location. This proves that having an oponent under direct fire, while being under indirect fire provides a tremendous advantage.

From the construction and analysis of the mathematical model of asymetric combat we were able to see that to defeat the insurgents , we must maintain:

$$y_o < \frac{\alpha {x_0}^2}{2\beta}$$

To keep this relation β must be the smallest possible, which is beyond the army's control. Hence, we have to focus in increasing the value of the fraction by properly increasing the value of the numerator.
Therefore, if we use:

$$x_0 = \sqrt{2} y_0 \Rightarrow \frac{\alpha {x_0}^2}{2\beta} = \frac{\alpha (\sqrt{2} y_0)^2}{2\beta} = \frac{2\alpha {y_0}^2}{2\beta} = \frac{\alpha {y_0}^2}{\beta}.$$

Hence to make sure that the value of the denominator increases, we must maintain the condition:

$$x_0 \geq \sqrt{2} y_0$$

That way it will possible to neutralize an insurgents' suppressive attack by keeping the numerical condition above [10]. This shows that for asymmetric combats under direct fire, the number of troops needed to withstand and neutralize the attach must be at least forty percent times greater than the number of insurgents taking part in the attack.

V. CONCLUSIONS

From the obtained calculation we can assert that mathematical models of combat serve as another weapon in the arsenal of the army. The advantage of the models is that they enable the battle field commanders to elaborate the strategy to use in combat, based on the number of needed number of troops and its firepower to neutralize the opponent.

A proper use of the square law of direct fire significantly helps to counter balance the technological advantage when fighting an army, smaller in size, but technologically superior. In practice many nations that lack advance military technology use the square law of fire, by choosing to increase their army's by at least three times the size of the opponent army.

The liner law of indirect fire provides a considerable advantage when having a greater firepower than a greater number of troops. This is because the probability to hit the enemy increases when firing as much as possible in the area where it is located.

The mathematical model of asymmetric combat, when under direct fire, provides a solution to counteract the enemy's significant threat. For this type of combat, it is decisive to count on with a numerical advantage that overcomes the insurgents by at least square root of two times its initial number as well as an optimal firepower. Otherwise, the risk of defeat is very likely.

The use of drone's technology allows to make smart transformations from indirect fire weapons to direct fire weapons, with the drones being the eyes of indirect fire weapons. This gives them a precise hit probability that otherwise will be reached with the use of a precise missile, whose single cost can easily exceed the cost of a hundred drones.

Though the application of the square law of direct fire and the linear law of indirect fire models can decisively help to plan the most optimal strategy and decision making, nonetheless the use of mathematical models in warfare is certainly limited by the unprediction of human behavior. As history shows the moral of troops, their willpower and motivation to fight are factors that a mathematical model does not take into consideration. Therefore, it would be beneficial to consider those factors to have a better understanding of the reality of combat and thus prevent unpredicted outcomes.

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