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Application of Interpolatory Methods of Model Reduction to an Elevated Railway Pier

Santiago Bertero, Mg. Ing.[1], Serkan Gugercin, Ph.D.[2], and Rodrigo Sarlo, Ph.D.[3]
[1]Universidad de Buenos Aires, Argentina, sbertero@fi.uba.ar
[1,2,3]Virginia Tech, United States, gugercin@vt.edu, sarlo@vt.edu

Abstract– ***Be it due to time constraints or insufficient processing power – or a combination of both – the use of models with large numbers of degrees of freedom (DoF) may be unsuitable to provide a client with results in a timely manner. The use of physics-based reduced models – or proxy structures – are popular among practitioners to solve this issue, as they keep intact all the underlying properties of the second order problems at a fraction of the cost.***

In this paper, interpolatory methods of model reduction are explored as an alternative, and applied to a 3D Space Frame. The methods chosen allow for structure-preserving reduced models and differ mainly on the selection of interpolation points. A comparison between the response of these reduced models and a proxy structure against two different types of inputs show that interpolatory methods are a viable, more flexible option when it comes to reducing the internal DoF's of a structural model, though engineering judgement helps to ensure it adequately captures the most relevant aspects of the response for the specific application.

Keywords—structured dynamics, reduced order modeling, interpolatory model reduction, H_2 norm, seismic.

I. Introduction

Model based, probabilistic structural design requires running complex models against a set of extreme and/or frequent dynamics events (e.g., ground motions, wind loading). The number of events that can be considered in current design protocols is primarily driven by the computational complexity of these models. Thus, efficient reduced order models have the potential to enable the simulation of a much broader set of events, leading to more robust probabilistic design. For civil structures, Modal Analysis (also known as Modal Truncation, MT) [1,2] is the most common approach in practice to perform model order reduction (MOR) and is available in commercial software such as SAP2000 [3]. However, this approach is typically limited to structures with classical damping, using equivalent modal damping for other cases (such as structures with added damping) or defaulting to direct-integration methods on the full model otherwise.

Application of projection-based MOR for the analysis of civil structures beyond modal analysis is not as common in the literature. In earthquake engineering, where structures typically exhibit significant nonlinear behavior, there have been different attempts to adapt established ROM strategies to the field, such as POD [20–22], though not exclusively, with other authors attempting to introduce approximations analogue to modal analysis [23,24]. In Ref. [4], reduced order modeling was used to design a controllable (active) damping system for a multistory building against wind loading. Projection-based MOR techniques that are rooted in the systems-and-control theory based on the concept of transfer function are commonly and successfully used in model reduction of dynamical systems. We refer the reader to Ref. [5–7] for a survey of such methods. These systems-theoretic techniques have been extended to model reduction of structured dynamics we consider in this paper; see, for example, Ref. [8–13] and the references therein.

One difficulty with projection-based MOR strategies is their intrusive nature. That is, they require access to the system matrices that govern the problem and may require modifications to the solver algorithms, which may not be possible with commercial software used in practice. Some of the projection-based techniques mentioned above (and employed in this paper) have been extended to the structured models we consider here; see, e.g., Ref. [14–19] and the references therein for some of the data-driven approaches to structured dynamics. However, this is not our focus in this paper and these considerations are left to future work.

Alternatively, the calibration of simplified models based on idealized behavior of complex structures are often used as they allow to stay in control of the physics, explain the results and shortcomings of the model, and work independently from the full model beyond using the calibration data. Examples for this can be found in Ref. [25], where nonlinear spring parameters are calibrated to represent the nonlinear stiffness matrix of the reduced model; and in Ref. [26] where simplified 1-DOF systems calibrated against the full model are used as proxy models as an intermediate step to evaluate the performance of a steel moment frame.

No matter the methodology, in order to maintain confidence in results reduced models should strive to preserve the physical structure of the system. In this work, we study the application of one such set of methods, namely interpolation-based MOR [5], to a real ground vibration case study on a linear system. The structure in question is an elevated railway pier, whose well-known response properties enable us to evaluate the physical relevance of a variety of reduced order models.

In future work, we envision scaling this approach to complex models for which engineering judgement may not be able to provide suitable assumptions for reduced modeling.

In structural dynamics, the response of a structure, modeled with discrete, lumped masses, to a time-varying input (e.g., wind loads, vehicle traffic, pedestrian load, earthquakes, etc.) can be written in the form

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$$M\ddot{x}(t) + G\dot{x}(t) + Kx(t) = Bu(t)\ ,\qquad y(t) = Cx(t)\ , \tag{1}$$

where $M, G, K \in \mathbb{R}^{n\times n}$ are symmetric matrices representing the mass, damping and stiffness of the structure respectively; $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are the input and output matrices; $x(t) \in \mathbb{R}^n$ is the displacement vector at each DoF; $y(t) \in \mathbb{R}^p$ is the output vector; and $u(t) \in \mathbb{R}^m$ is the input.

The internal degrees of freedom (DoF's) n can sometimes become too large to evaluate the response against a wide variety of inputs (e.g., when performing probabilistic analysis) and makes the process computationally prohibitive. The goal of Model Order Reduction (MOR) is to circumvent this issue by building a model with only r DoF's where $r << n$:

$$M_r\ddot{x}_r(t) + G_r\dot{x}_r(t) + K_rx_r(t) = B_ru(t)\ ,\qquad \tilde{y}(t) = C_rx_r(t)\ , \tag{2}$$

such that the difference between $y(t)$ and $\tilde{y}(t) \in \mathbb{R}^p$ is small for a wide range of input choices.

During design, the most common method to achieve this is by simplifying the mathematical description through further idealizations or assumptions about the underlying physical behavior of the structure. Note that this does not (only) refer to the use of a coarser discretization in a Finite Element Model, but a significant change in modeling. For example, it is typical to assume either a fully rigid or fully flexible floor diaphragm for the design of the lateral system in a building [27], or to simplify the modeling of the deck of a bridge using a grid analogy [28], both of which significantly remove the number of DoF's to consider. Similarly, for nonlinear models, concentrated plasticity models are typically used over fiber (distributed) models to reduce computational time [29].

The advantages of this approach to creating a reduced model is that the reduced matrices $M_r, G_r, K_r \in \mathbb{R}^{r\times r}$ maintain all the properties of the full model (namely, being symmetric and positive definite), and that the model itself then holds physical meaning which helps with explainability. However, building reduced models based on engineering judgement and the introduction of further simplifications – referred to here as proxy structures – requires experience, expertise and good insight into the internal mechanisms of the structure. Moreover, it is an ad-hoc solution, meaning that for each problem and for each degree of simplification an entirely new model needs to be developed. It may also not be possible to create accurate proxy models for more complex structures (such as buildings with floorplan irregularities or with combined structural systems).

Another popular way of evaluating the linear response of structures with classical damping is via modal analysis, selecting only a subset of modes (also known as Modal Truncation, MT). The main advantage of this method is that the resulting model in modal coordinates leads to an uncoupled system of equations, from which the response in the physical coordinate statement can be obtained via superposition [2]. For the non-classically damped case, where mode-shapes become complex, modal analysis is typically replaced by time-stepping methods (response history analysis) using the full order model, or by approximating the damping matrix to a classical formulation that leads to similar modal damping ratios for use in modal analysis. Modal Truncation is a particular example of a projection-based model reduction; the dynamics are projected based on the eigenvalues and mode shapes to be retained in the reduced model.

A second order dynamics of dimension n as in Equation (1) can be converted to the first-order form of dimension $2n$ by redefining the state variable as

$$z(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \in \mathbb{R}^{2n}\ , \tag{3}$$

which then leads to the first-order state-space form

$$E_2\dot{z}(t) = A_2z(t) + B_2u(t)\ ,\qquad y(t) = C_2z(t)\ , \tag{4}$$

where

$$E_2 = \begin{bmatrix} I_n & 0 \\ 0 & M \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & I_n \\ -K & -G \end{bmatrix},\quad B_2 = \begin{bmatrix} 0 \\ B \end{bmatrix},\ \text{ and } C_2 = \begin{bmatrix} C & 0 \end{bmatrix}. \tag{5}$$

Note that this state-space representation is the same one used to analytically obtain the damping ratio of structures with non-classical damping (e.g.: structures with added damping [30–32]). However, even though it is an exact representation of the second order problem, it is not guaranteed that the reduced model from performing projection-based MOR on Equation (4) preserves the structure. In other words, it may not be possible to write it in second-order form, obscuring the underlying physics of the problem. Recent developments have resulted in methods that allow for structure-preserving model reduction, which are be the focus of this paper.

II. Structure-Preserving Interpolatory Methods of Model Reduction

Interpolatory methods of model reduction are a subset of methods that perform model reduction via projection [5] and they can be directly applied to the second-order form of Equation (1). The main goal of structure-preserving MOR of the second-order system is to find two model reduction bases $V_r, W_r \in \mathbb{R}^{n\times r}$ so that the reduced model quantities in Equation (2) are obtained as

$$M_r = W_r{}^TMV_r,\ G_r = W_r{}^TGV_r,\ K_r = W_r{}^TKV_r,\quad B_r = W_r{}^TB \ \text{ and }\ C_r = CV_r\ . \tag{6}$$

Note, then, that the reduced model is of the same form as the full model in Equation (1) and keeps its second order nature. Moreover, if W_r is chosen equal to V_r, then the reduced square matrices will stay symmetric, a crucial property in structural analysis based on the First Law of Thermodynamics. Another thing of note is that, if the full model had proportional damping, i.e.,

$$G = \alpha M + \beta K \ , \qquad (7)$$

then the reduced model will also have proportional damping with coefficients α and β. Proportional damping, an idealization of the unknown damping properties of a structure, is nonetheless a common approximation of the problem [2]. For other structure-preserving projection-based MOR for second-order systems we consider there, we refer the reader to [8].

Interpolatory methods for MOR amount to choosing the model reduction matrices V_r and W_r such that the transfer function of the reduced model from Equation (2), i.e.,

$$H_r(s) = C_r(s^2 M_r + sG_r + K_r)^{-1} B_r \ , \qquad (8)$$

interpolates the transfer function of the full model from Equation (1), i.e.,

$$H(s) = C(s^2 M + sG + K)^{-1} B \ , \qquad (9)$$

at a number of interpolation points $s_i \in \mathbb{C}, i = 1,2,\dots,r$. However, when dealing with multiple inputs and outputs, the transfer function $H(s)$ is matrix valued and full interpolation of the transfer function matrix is usually not performed. Instead, tangential interpolation is used along the right directions $b_i \in \mathbb{C}^p, i = 1,2,\dots,r$ and left directions $c_i \in \mathbb{C}^m, i = 1,2,\dots,r$. Even though higher-order interpolation is possible, for simplicity of the presentation, assume that we only want to enforce interpolation up to the first derivative. Therefore, given the interpolation points s_i, the right-directions b_i, and the left-directions c_i, the goal is to construct V_r and W_r such that the reduced transfer function $H_r(s)$ satisfies the following interpolation conditions:

$$H(s_i)b_i = H_r(s_i)b_i, \ \ {c_i}^T H(s_i) = {c_i}^T H_r(s_i), \text{ and} \qquad (10)$$
$${c_i}^T H'(s_i) b_i = {c_i}^T {H_r}'(s_i) b_i \ . \qquad (11)$$

Define $\varkappa(s) = s^2 M + sG + K$. Then, the interpolation conditions in Equations (10,11) can be enforced by choosing the model reduction bases V_r and W_r as

$$V_r = [\varkappa(s_1)^{-1} B b_1 \cdots \varkappa(s_r)^{-1} B b_r], \text{and} \qquad (12)$$
$$W_r = [\varkappa(s_1)^{-T} C^T c_1 \cdots \varkappa(s_r)^{-T} C^T c_r] \ . \qquad (13)$$

Assuming that symmetry be retained in the reduced model, one usually chooses $W_r = V_r$. In that case, only the right tangential interpolation conditions are enforced, i.e., $H(s_i)b_i = H_r(s_i)b_i$ for $i = 1,\dots,r$. Interpolation points may be complex. However, the underlying structural model is real. In that case, it can be ensured that V_r stay real by selecting a complex conjugate pairs of interpolation points and tangent directions. For details we refer the reader to Ref. [5]. Of course, the main question – as with any interpolatory method – is how to select the interpolation points and, in the MIMO case, the tangential directions.

A. Interpolation Point Selection Based on Exact Condenser Distribution

Though the derivation exceeds the scope of this paper, Ref. [33] proposed an interpolation point selection in the SISO case for second order problems with proportional damping. In essence, if the damping matrix G is defined via proportional damping with coefficients α and β as in Equation (7)), then Ref. [33] proposes choosing

$$s_0 = \sqrt{\frac{\alpha}{\beta}} \qquad (14)$$

as the interpolation point given an exact condenser distribution. In this case, as opposed to choosing multiple interpolation points and imposing the interpolation as in Equations (10,11) up to the first derivative, one chooses the single interpolation s_0 and matches $H(s)$ and its first $r-1$ derivatives at $s = s_0$.

B. Extension of IRKA to Second-order Dynamics

Iterative Rational Krylov Algorithm [34] (IRKA) is an iterative method for the selection of locally optimal interpolation points in the H_2 norm when reducing dynamical systems given in the first-order form as in Equation (4). It can be shown that for MOR of the first-order dynamical systems from Equation (4) in the H_2 norm, the optimal interpolation points si correspond to the mirror images of the reduced-order poles across the imaginary axis and the optimal directions $b_i \in \mathbb{C}^p$ and $c_i \in \mathbb{C}^m, i = 1,2,\dots,r$ correspond to residue directions of the resulting reduced model. The process then becomes iterative as the poles of the reduced model depend on the interpolation points chosen. Then, the selected interpolation points and directions are repeatedly updated based on the reduced model obtained until convergence is reached. We refer the reader to Ref. [5, 34] for details.

In applying IRKA to a second order problem, however, the following issue arises. A second order system with r DoF's has $2r$ poles. Then, if the iteration starts with r interpolation points, the second iteration would have $2r$ points, the third $4r$ and so on, resulting in an exponentially increasing model order with each step. To remedy this, Ref. [35] proposed two different methods for the selection of interpolation points. Neither of these methods satisfies the optimality conditions as IRKA does for first-order systems, but they still provide accurate, high-fidelity approximants.

1) SO-IRKA: In SO-IRKA, only a subset of size r from the full set of $2r$ poles is used for the selection of the new interpolation points. The selection method can be, for

example, those that are closest to the imaginary axis. In a structural analysis problem, that means the poles with smallest damping, which in turn typically means the lowest natural frequencies when Rayleigh Damping is used.

The process, then, can be summarized in the following way:

1) Set r interpolation points si and tangential directions bi.
2) Construct V_r as in Equation (12). Choose $W_r = V_r$
3) Build the reduced model matrices as in Equation (6)
4) Find the $2r$ poles λ_i and right residues b_i of the reduced model. This can be done by writing the equivalent first order problem as in Equation (4)
5) Set the new interpolation points as $s_j = -\lambda_j$, keeping only the r poles closest to the imaginary axis, and set the new directions b_j, the associated right residues
6) Repeat until convergence of the interpolation points/reduced model poles

2) SOR-IRKA: In SOR-IRKA, a second layer of model reduction is introduced. Once the order r reduced model is built, the selection of the new interpolation points is done through a second model reduction. Given the r-sized second-order model, the equivalent first order model is reduced from $2r$ to r by any method of choice, and the poles of this new r-sized first order problem are used for the definition of the new samples.

There are a couple of points to note here. One is that, though applying model reduction to the equivalent first order problem may destroy structure, this is only done for the purpose of finding the interpolation points, and the end result of the process is still a second order reduced model. The other is that any method of choice is viable to perform the reduction from $2r$ to r such as IRKA [34] or Balanced Truncation [36,37]. The process, then, follows the following steps:

1) Set r interpolation points s_i and tangential directions b_i.
2) Construct V_r as Equation (12). Choose $W_r = V_r$
3) Build the reduced model matrices as in Equation (6)
4) Write the equivalent first order problem as in Equation (4)
5) Reduce the first order problem from size $2r$ to r. IRKA, Balanced Truncation, or other MOR methods are all viable options for this.
6) Find the r poles λ_j and right residues b_j of the reduced first order model
7) Set the new interpolation points as $s_i = -\lambda_j$ and set the new directions b_i, the associated right residues
8) Repeat until convergence of the interpolation points/reduced model poles

For application of SO-IRKA and SOR-IRKA in damping optimization, we refer the reader to [38]. We also note that in this paper it is assumed that the linear system in the construction of V_r in Equation (12) are solved via direct methods. For the impact of using iterative/inexact methods on the interpolation, see Ref. [39].

III. Case Study: Elevated Rail Pier During The Construction Stage

In this section, the different methods discussed will be applied to the piers of an Elevated Railway in Buenos Aires, Argentina (Fig. 1a). Built in 2018, it was the first of its kind in the country. Spanning over 100km, the Elevated Railway was constructed above the existing railroad. It is out of the scope of this paper to discuss the details of this project, which involved in-situ testing to calibrate a numerical model, as well as the evaluation of different acceptance criteria in order to increase the speed of construction without incurring in added risk of cracking the concrete piers due to early-age loading. More information about the project can be found in [40].

The full model for the 3D-frame consists of a $n = 102$ DoF's, lumped-mass, second order system. It has $m = 3$ inputs acting equally at both supports:

1) $u_1(t)$: Ground acceleration in the plane of the frame
2) $u_2(t)$: Ground acceleration perpendicular to the frame
3) $u_3(t)$: Ground acceleration in the vertical axis

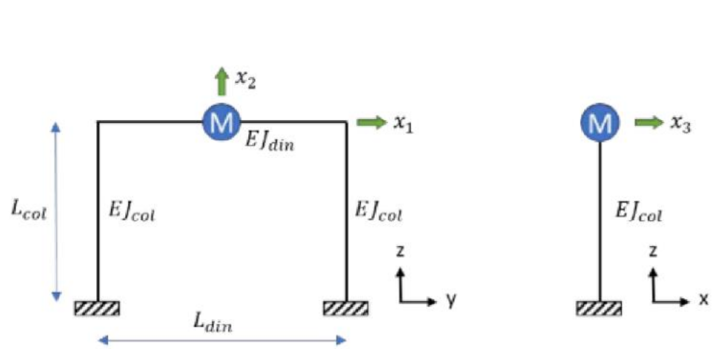
and it has $p = 6$ outputs that were deemed of interest:

1) $y_1(t)$: In-plane moment at the base of the column
2) $y_2(t)$: Out-of-plane moment at the base of the column
3) $y_3(t)$: Out-of-plane moment at beam's end
4) $y_4(t)$: In-plane moment at beam's end
5) $y_5(t)$: In-plane moment at beam's midspan
6) $y_6(t)$: Out-of-plane moment at beam's midspan

All the matrices were developed following the methodology explained in Ref. [42,43]. The damping matrix G was constructed assuming proportional damping, and setting the damping coefficient of the first two modes of the structure as 1%. Meanwhile, the internal DoF's include rotational ones, which in a lumped mass model carry no mass. Then, the mass matrix M for the problem is singular. This, in turn, converts the system into a set of Differential Algebraic Equations (DAEs), for which interpolation theory can still be applied [41].

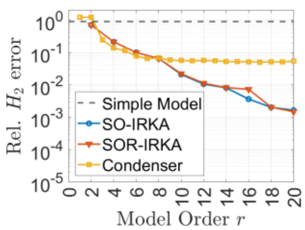
At the time, time constraints and the goal of having a controllable, simple model of the structure for the analysis resulted in the development of a simplified model based on engineering judgement of order $r = 3$ to serve as a proxy for the expedited structural evaluation during the project. In order to reach that model, several additional assumptions were introduced, such as axial rigidity of the frame elements, neglecting out-of-plane deformation of the structure beyond that of the columns, and the exploitation of symmetry and anti-symmetry properties. The mass matrix (which for this uncoupled model represents mass participation) was then adjusted to match the natural frequencies of the main modes of the structure. This approach was chosen in favor of applying

(a) As-built structure

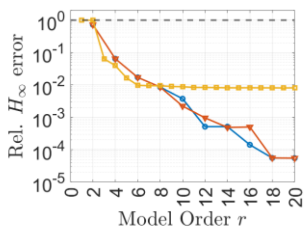


(b) Assumed behavior for the development of the proxy structure

Fig. 1 Sample pier evaluated during construction.



(a) H_2-norm



(b) H_∞-norm

Fig. 2 Relative error of the reduced models

Rayleigh's Method with deformed shapes based on the assumptions above since the data was already available from the testing phase. A schematic of the assumed behavior of the proxy structure can be seen in Fig. 1b.

In the following subsections, we'll evaluate the accuracy of such model and compare it to the result of applying the interpolatory methods discussed in Section II.

A. *Accuracy in Terms of the H_∞ and H_2 Norms*

In comparison to the proxy structure described before, interpolatory model reduction was performed on the full model by applying the (a) Optimal Single Interpolation Point Method, (b) SO-IRKA and (d) SOR-IRKA (using Balanced Truncation in the intermediate step). For all methods, the model order r was defined in the range 0 to 20. As both SO-IRKA and SOR-IRKA deal with complex conjugate interpolation points, only even numbered orders were obtained.

As mentioned before, the full model had a singular M matrix, which in turn means that the E_2 matrix in the first order representation from Equation (4) is also singular. For DAEs, applying IRKA without modifications does not guarantee H_2 optimality. In fact, the error in the reduced model may grow unbounded as $s \to \infty$. However, in this specific problem, due to the specific choices of B and C matrices, we have $\lim_{s\to\infty} H = 0$. Thus, we will apply all three methods as is, obtain a reduced model with $\lim_{s\to\infty} H_r = 0$, and will compute the error norms appropriately. Though not shown, this property was verified for this paper, the results being consistent with known behavior of structures under dynamic loading (displacements tend to zero as the frequency of excitation tends to infinity).

The relative errors in both the H_∞ and H_2 norms are shown in Fig. 2. Both plots lead to similar conclusions. The use of a single interpolation point leads to comparable errors to the methods using multiple interpolation points (SO-IRKA and SOR-IRKA) for low model orders ($r < 8$).

However, its performance suffers for higher orders. Further analysis showed that the singular optimal point interpolation, as the order increased, was including new poles at very high frequencies (above 100Hz) in the resulting reduced model, which in this problem are both highly damped and likely not to be excited by typical loading conditions though are important in higher order derivatives. Meanwhile, SO-IRKA and SOR-IRKA stayed mostly within the 0Hz to 100Hz range, which is where the bulk of the input will be set in structural applications. This is due in part because the initial interpolation points were set in that range (knowing the range of interest), even if the iterative process can lead the final poles to be outside of the range. It is important to remember here that IRKA finds a locally optimal approximant.

When it comes to SOR-IRKA and SO-IRKA, no discernible difference can be seen and both methods seem to perform equally well. Given that the results are similar between the two – something also seen in Ref. [35] in most of the examples presented – there is no visible impact in adding an intermediate step in the pole selection process for this structure. As such, for future comparisons in this report, SO-IRKA will be the focus over SOR-IRKA and the single interpolation point method.

B. *Comparison of Select Transfer Functions*

Next, the transfer functions for each of the models will be compared, as they provide good insight into the response of different models. Since this is a MIMO system, select terms of the transfer function will be analyzed. These were specifically chosen as they best represent all the information available about the structure, as well as allowing to see what each reduced model is able to capture about the full model.

The following three transfer functions were selected (Fig. 3):

- $H_{6,2}$: Out-of-plane moment at midspan of the beam due to out-of-plane ground motion
- $H_{4,3}$: In-plane moment at the end of the beam due to vertical ground motion
- $H_{1,1}$: In-plane moment at the base of the column due to in-plane ground motion

For the proxy model used in 2018, it can be seen that it was built to match the response at lower frequencies (the fundamental mode in each "direction" of analysis). However, as one of the assumptions was that there was no out-of-plane deformation beyond that of the columns, it predicts no out-of-plane moment at the end of the beam, which can be seen on Fig. 3a. Although that is a large source of error in terms of the norms, the choice of $H_{6,2}$ was made due to the expectation that these moments would end up being small enough for typical inputs, something that will be evaluated in the following subsections. Another point to note is that, since the model only covers the fundamental modes, it could lead to the underestimation of bending moments (especially at the base of the columns) if the input contains energy at higher frequencies, as evidenced in Figs. 3b and 3c.

As for the reduced model obtained using SO-IRKA, Fig. 3 provides details on how the chosen model order affects the results. First, Fig. 3a shows that the out-of-plane response is being captured by the smallest model evaluated ($r = 2$), which in effect is the only portion of the response included in the model, ignoring the in-plane and vertical ground motion effects. Increasing the model order adds upon previous results. Looking at Fig. 3c, it is at order $r = 4$ that the in-plane response is captured in the transfer function, with the out-of-plane response remaining equal (Fig. 3a). Interestingly, it does so across the frequency band, and not just the first fundamental mode, which means that the estimate of the in-plane moments should improve over the proxy model, especially for high frequency excitations. Similarly (see Fig. 3b), it is not until order $r = 6$ (double the proxy's order) that the reduced model obtained via SO-IRKA is able to capture the vertical response of the structure and, in turn, fully capture the most important properties of the frame's response (i.e., the three fundamental modes described by the proxy model). That means that, even if the H_2 and H_∞ errors were much smaller than for the proxy, from a structural engineering standpoint using $r = 2$ and $r = 4$ does not result in acceptable reduced models. However, using $r = 6$ leads to a model that does a reasonable job matching the transfer function of the full model while capturing the important dynamic properties from a design standpoint. As a final note, SO-IRKA with order $r = 12$ is the smallest order that manages to closely match the response across the frequency band of interest, with the largest discrepancies occurring in the 80 to 100Hz (Fig. 3b).

C. Simulation Against a Local Earthquake

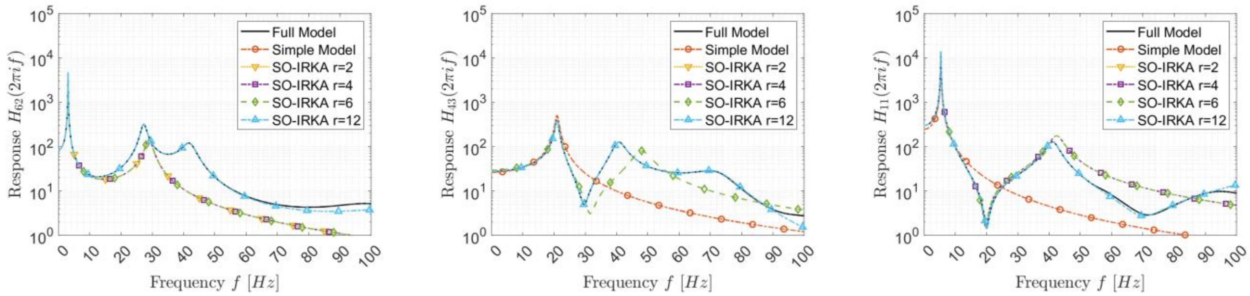
In order to see how these error measures translate to errors when evaluating the structure, the models were subjected to two different types of inputs. The first of these is the measured ground motion during the 2018 Buenos Aires Earthquake. This earthquake had its epicenter 45km away from Villa Ortuzar and had a magnitude M_w of 3.8 [44,45]. Being a local earthquake, its frequency bandwidth is larger than usual, though still quite small compared to human-induced vibrations. In Fig. 4, both the time history and its Fourier Transform can be observed. The bulk of the energy is concentrated below 10Hz, which will excite only the fundamental in-plane and out-of-plane modes of the structure.

The results of the simulation can be seen in Table 1. Presented in the table are the maximum absolute values of each output. This is different from what the reduced models are trying to approximate (the full time history of each output), but for design purposes it was deemed appropriate. This is especially true when considering that the main concern was the possibility of the structure cracking, which can be assumed happens when at any given time the bending moment goes over the cracking moment. In fact, if that were to happen the system would no longer be linear in the first place. One could consider replacing the C matrix with the max operator and set the output as simply the maximum value of the bending moments, but that would not correspond to the linear output structure we consider here. Another option could be simply to choose $C = I$, the identity matrix. Instead, we used the C matrix defined above and assumed that if a model can predict the time history accurately enough, then the peak values would also be accurate.

The proxy model exhibits adequate performance, capturing the essential outputs and predicting their value. The maximum error is 0.62kNcm for the in-plane moment at the base of the column (y_1), a 13% difference. Since earthquakes generally mostly excite the fundamental modes, this was an expected outcome. As mentioned before, the order $r = 2$ model obtained with SO-IRKA is not suitable, as it fails to capture the in-plane response (y_1, y_4, y_5), which is of utmost importance. Note, however, that it almost perfectly matches all the outputs related to the out-of-plane motion of the structure (y_2, y_3, y_6). As mentioned before, the order $r = 4$ model incorporates the in-plane motion response and from then on all the models result in almost negligible error. Note that, since vertical earthquake loads tend to be smaller than their horizontal counterparts, and the input is low frequency, the vertical response is of little significance (exemplified by y_5).

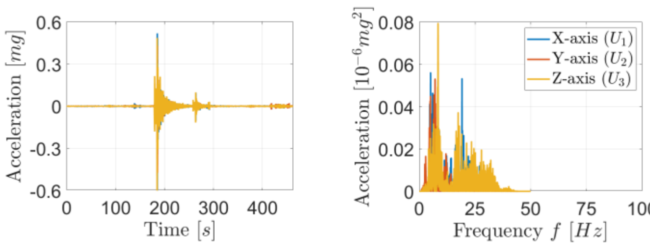
D. Simulation Against a Train Passing

Now, the same analysis is done for the vibrations induced by a train passing by at close proximity. This time history was captured on site and served as a final check for the original



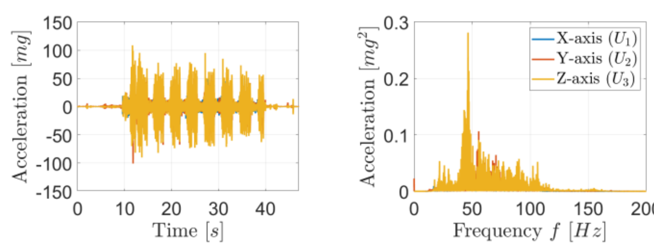
(a) $H_{6,2}$ (b) $H_{4,3}$ (c) $H_{1,1}$

Fig. 3 Comparison of selected transfer functions.



(a) Time history (b) Fourier Transform

Fig. 4 2018 Buenos Aires earthquake.



(a) Time history (b) Fourier Transform

Fig. 5 Measured ground motion induced by a train passing by at close range

TABLE 1

Results summary for Earthquake input

Moment $[kNcm]$	$max\lvert y_1(t)\rvert$	$max\lvert y_2(t)\rvert$	$max\lvert y_3(t)\rvert$	$max\lvert y_4(t)\rvert$	$max\lvert y_5(t)\rvert$	$max\lvert y_6(t)\rvert$
Full Model	4.63	2.02	0.06	3.61	0.79	0.42
Proxy Model	4.01	1.81	0.00	3.55	1.00	0.00
SO-IRKA r=2	0.00	2.02	0.06	0.00	0.00	0.41
SO-IRKA r=4	4.66	2.02	0.06	3.62	0.00	0.41
SO-IRKA r=6	4.63	2.02	0.06	3.60	0.79	0.41
SO-IRKA r=12	4.63	2.02	0.06	3.61	0.79	0.42

TABLE 2

Results summary for Train input

Moment $[kNm]$	$max\lvert y_1(t)\rvert$	$max\lvert y_2(t)\rvert$	$max\lvert y_3(t)\rvert$	$max\lvert y_4(t)\rvert$	$max\lvert y_5(t)\rvert$	$max\lvert y_6(t)\rvert$
Full Model	5.58	2.08	0.36	4.67	14.14	2.88
Proxy Model	2.15	0.42	0.00	4.23	14.76	0.00
SO-IRKA r=2	0.00	3.00	0.16	0.00	0.00	1.17
SO-IRKA r=4	3.66	3.00	0.16	1.69	0.00	1.17
SO-IRKA r=6	7.88	3.00	0.16	4.68	11.80	1.17
SO-IRKA r=12	5.55	2.14	0.36	4.67	14.13	2.85

project. Both the time history data and the frequency domain information can be seen in Fig. 5.

At a quick glance the differences between a train input and an earthquake are evident. First of all, a local earthquake mostly consists of one big shock and the response of the structure is dominated by its impulse response. That is why most of the response is explained by the fundamental modes. Meanwhile, train-induced vibrations contain energy at high frequencies, namely in the 40Hz to 100Hz range, and so higher order modes will be excited and will explain a significant portion of the total response. It is expected then that the proxy model would perform worse under these conditions, and generally speaking this is a difficult input to approximate.

The results of the simulations can be seen in Table 2. In this case, vertical motion is the dominant input, and the proxy model does a very good job of capturing the response in terms of y_5 (4% relative error). However, as expected, it fails to accurately measure the bending moments at the base of the column due to ignoring the effects of higher order modes with an absolute error of 3.43kNm at y_1. Note how, in this case, the models obtained with IRKA at order $r = 2$ and $r = 4$ are not suitable for this type of loading, as they do not include the vertical response at all. That also extends to other outputs, namely y_4 (in-plane moment at beam's end) since it depends on both the in-plane motion and the vertical motion. At order $r = 6$, which is the first order to include all relevant aspects of the structural behavior, reduces the maximum absolute error to 2.30kN m (at y_1) and is overall an improvement over the proxy model already. This is due to the fact that it does include in its transfer function an estimation of the response at higher frequencies. Finally, as before it can be seen that by order $r = 12$ the result is almost a perfect match to the full model.

From above it follows that the model obtained by SO-IRKA can be improved by simply selecting a different model order. However, improving the Proxy Model would be a very difficult task. Every idealization made would have to be reviewed, leading to the development of a new model based on a different set of assumptions. Moreover, trying to introduce higher order modes into a simplified model can prove to be exceptionally difficult as they naturally trend towards capturing the fundamental frequencies.

IV. CONCLUSIONS

In this paper, different strategies were evaluated for the construction of a reduced model that accurately represents the behavior of a 3D Space Frame under a variety of inputs. Both proxy models built on engineering judgement and reduced models based on interpolatory methods were studied. Results show that the former have a place in structural engineering, as they allow for low DoF approximations that are sufficiently accurate (especially for low frequency inputs) and manage to capture the most important aspects of a structure’s response when derived properly. That last aspect is essential, though. Building proxy models is a difficult task that requires experience, knowledge and deep understanding of the fundamental behavior of a given structure. Each model is a unique solution for a unique problem and, as such, can be extremely costly to develop.

Projection-based model reduction, meanwhile, proves to be a viable alternative. In its interpolatory framework, it allows for structure-preserving reduced models that hold physical meaning and can be effectively interpreted as smaller structural systems, which helps when it comes to communicating with peers and clients. What is more, its systematic approach to model reduction means that it is possible to increase precision at the low (relatively speaking) cost of re-running a given algorithm. This flexibility is of upmost importance when looking to maximize the accuracy-to-cost ratio, as shown in the results of the case study for two different loading scenarios (Tables 1 and 2).

It should be noted that in the case study shown, the value of reduced order modeling is limited, since the full model was not computationally intensive to run. However, it serves to highlight and build confidence on the intuitive performance of interpolatory methods. As model complexity increases, such models allow a systematic approach to tuning the model complexity to the desired accuracy, all while maintaining physically relevant behavior at a fraction of the computational cost. Nevertheless, it is important to cross-reference these reduced order models with simplified engineering to avoid unexpected results and ensuring that the response is aligned with the expected types of loading.

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