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Optimal Exponentially Weighted Moving Average Of T^2 Chart

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Abstract

We propose a chart for normal variables that uses a statistic that is a recursive form of Hotelling's T^2 statistic. The objective of this study is to improve the performance of the T^2 chart. To analyse the performance of this chart, we have developed a user-friendly program that finds the best parameters through Genetic Algorithms. This algorithm minimises the out-of-control ARL (Average Run Length) for a proposed shift in the mean vector under the restriction of a desired in-control ARL value. The proposed control chart shows better performance than the Hotelling T^2 chart.

Keywords: Hotelling T^2 control chart; Optimisation; Normal Variables; Genetic Algorithm; Average Run Length.

1. INTRODUCTION

It is a permanent aspiration to improve the performance of the T^2 control chart, which was created to detect abnormal fluctuations in a normal multivariate production process. Many companies apply this chart due to its ease of use and interpretation; however, it is limited in terms of the information it provides.

Hotelling's T^2 control chart is a tool that detects significant changes at specific times rather than small changes that occur over time. Currently, due to the automation processes, the speed of production has increased, which makes it dangerous not to detect these small changes; doing so would allow timely actions to be taken with minimal impact for the organisation.

It is now more necessary to have control charts that detect small changes that occur during the production process. The first univariate versions proposed by Page [1] (CUSUM: Cumulative Sum Control Chart) and Roberts [2] (EWMA: Exponentially Weighted Moving Average Chart) were improved by their multivariate versions developed by Lowry et *al.* [3] (MEWMA: Multivariate Exponentially Weighted Moving Average chart) and Crosier [4] (MCUSUM: Multivariate Cumulative Sum Control Chart).

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At the end of the 20th century, new proposals were made in the multivariate control charts to optimise their performance. Many authors have contributed in this sense: 'Control chart for multivariate attribute processes' [5], 'Monitoring and Control of a Normal Multivariate Process' [6], 'The Variable Dimension T^2 Control Chart' [7], 'Optimum Multiple and Multivariate Poisson Statistical Control Charts' [8], 'Control charts with variable dimensions for linear combinations of Poisson variables' [9], 'One-sided cumulative sum control chart for monitoring shift in the scale parameter delta (δ), of the new Weibull-Pareto distribution'[10], 'Optimum variable-dimension EWMA chart for multivariate statistical process control' [11], 'Variable Parameter chart' [12], and among others.

These proposals and many others have improved the parameters of the control charts or combined traditional control charts and were created to optimise the resources used in the control of the process. However, each of them strengthens certain cases and presents weaknesses in others, which makes them sensitive to permanent improvements.

For this reason, this study proposes a new control chart that combines the ability to detect small changes in an EWMA chart using the multivariate scheme of the Hotelling T^2 control chart; that is, an EWMA version is proposed for a T^2 control chart, an EWMAT^2 control chart, which, under certain conditions, improves the performance of the preceding control chart.

2. ANALYSED CONTROL CHARTS

This section describes the theoretical basis of the proposed multivariate control chart, the EWMAT^2 chart. Moreover, the Hotelling T^2 and MEWMA charts are briefly described.

The ARL (average run length) has been used to evaluate the response of the proposed chart to changes in the mean vector of the process. The ARL is defined as the average number of samples in the control chart until the statistic of control takes a value outside the control limits. It is desirable to have high values of ARL when the process is in-control; in these cases, it is denoted as ARL_0. If the process is out-of-control, the chart should detect this quickly, and the value of the ARL (ARL_1) in these cases should therefore be as small as possible.

If the statistic to be used in the control chart is independent of time, the ARL behaves like the mean of a geometric random variable, and it must be calculated as ARL=1/P (P represents the probability that the statistic takes a value out of the control limits of the chart). However, if the statistic depends on its values in the past, there are other procedures to calculate the ARL values such as Markov chains, simulation, or integral equations [13]. This study used a Markov chain to calculate the ARL for the EWMAT^2 chart.

2.1 The Hotelling T^2 Control Chart

The Mahalanobis distance measures the closeness between two p-dimensional random variables (X, Y) with equal distribution function and variance and covariance matrix [14]:

$$d_m(X,Y) = \sqrt{(X-Y)^T\Sigma^{-1}(X-Y)} \quad (1)$$

Let $\boldsymbol{X^T} = (X_1, X_2, \dots, X_p)$ be a set of variables with mean vector $\mu_0{}^T = (\mu_{01}, \mu_{02}, \dots \mu_{0p})$ and variance and covariance matrix, Σ, then the square of the Mahalanobis distance for the set of variables $\boldsymbol{X}$ is:

$$d_m^2(X,Y) = (X-\mu)^T\Sigma^{-1}(X-\mu) \quad (2)$$

If $\boldsymbol{X}$ has a multivariate normal distribution, $X \sim N_p(\mu_0, \Sigma)$, then the χ_p^2 statistic has a chi-square distribution with p degrees of freedom:

$$\chi_p^2 = n(\overline{\boldsymbol{x}} - \mu_0)^T\Sigma^{-1}(\overline{\boldsymbol{x}} - \mu_0) \text{ (3)} \quad (3)$$

where $\overline{\boldsymbol{x}}$ is the sample mean vector and n is the sample size.

When the parameters of a process are unknown, it is necessary to estimate them. Let S and $\widehat{\boldsymbol{\mu}}$ be the estimators of the variance and covariance matrix and the mean vector, respectively. The T^2 statistic was proposed by Hotelling[15] and is commonly used for the multivariate control of variables with multivariate normal distribution:

$$T^2 = n(\overline{\boldsymbol{x}} - \widehat{\boldsymbol{\mu}})^{\boldsymbol{T}}\boldsymbol{S^{-1}}(\overline{\boldsymbol{x}} - \widehat{\boldsymbol{\mu}}) \quad (4)$$

The distribution of the T^2 statistic converges to a chi-square distribution with p degrees of freedom when the number of samples needed to calculate S and $\widehat{\boldsymbol{\mu}}$ tends to infinity [16]. Considering this, the T^2 control chart requires an upper control limit that can be defined by $UCL = \chi_{\alpha,p}^2$, where α is the level of significance and p is the number of monitored variables of quality.

When the process has a deviation in at least one of the means of the quality variables, then the vector $\boldsymbol{\mu_1}$ moves away from the in-control vector of means $\boldsymbol{\mu_0}$ at a Mahalanobis distance of d. In this case, the T^2 statistic has a non-central chi-squared distribution with p degrees of freedom and non-centrality parameter:

$$\lambda = nd^2 = n(\boldsymbol{\mu_1} - \boldsymbol{\mu_0})^T\boldsymbol{\Sigma^{-1}}(\boldsymbol{\mu_1} - \boldsymbol{\mu_0}) \quad (5)$$

For this control chart, the ARL is obtained by:

$$ARL = \frac{1}{1 - P(T^2 < UCL)} \quad (6)$$

If the statistic takes a value greater than or equal to the upper control limit *(UCL)*, the T^2 chart presents a signal.

2.2 The EWMA of Hotelling T^2 Control Chart

We propose an EWMA version of T^2 to improve the performance of T^2 chart. Its statistic is:

$$EWMAT^2{}_t = r * (T^2)_t + (1-r) * EWMAT^2{}_{t-1} \quad for\ t = 1, 2, \dots \quad (7)$$

Where $(T^2)_t$ is the value of the T^2 statistic calculated using equation (4) with the values of the t-th sample, r is the smoothing constant that complies with the constraint of $0 < r \leq 1$ (with the particularity that if $r = 1$, the $EWMAT^2$ control chart would result in the same T^2 chart).

The initial value of the statistic is $EWMAT^2{}_0 = E(T^2) = p$, which is the in-control mean of the T^2 statistic, that is, the mean of a central chi-squared random variable with p degrees of freedom. The EWMAT^2 requires only a control limit, the upper control limit, *UCL*.

As previously mentioned, due to the dependence of the statistic on its previous value, a Markov chain has been used for the calculation of the ARL. We must also consider that when we use control charts with memory effect, we need an initial value to begin the recursive calculations. In addition, ARL_1 can be different for the case in which the process is out-of-control from the beginning (*zero-state* ARL_1) or for the case in which the process starts as in-control and the shift subsequently occurs (*steady-state* ARL_1).

The procedure for the calculation of the ARL through a Markov chain used in this research has also been considered by other authors [17, 18, 12,13].

For the calculation of the ARL, let us divide the interval between *0* and *UCL* into m groups, and denote the upper and lower limits of the i-th interval as U_i and L_i for i = 1,2,..., m. We also define M_i as the midpoint of the *i*-th interval. The width of each interval would be given with the following expression:

$$h = \mathrm{UCL}/\mathrm{m} \tag{8}$$

The expressions for the upper limit, lower limit, and midpoint of the *i*-th interval would respectively be the following for *i* =1, 2,…, *m*

$$U_i = i * h \tag{9}$$

$$L_i = (i-1) * h \tag{10}$$

$$M_i = (2i-1) * (h/2) \tag{11}$$

Let us define a Markov chain with *m* transient states such that *i*-th state corresponds to the event in which the $EWMAT^2$ statistic is in the *i*-th sub-interval. In addition, an absorbent state would be the case in which the $EWMAT^2$ statistic takes a value greater than or equal to UCL.

Let us also define the transition matrix Q with an m x m dimension, which is formed by the probabilities q_{ij} that the $EWMAT^2$ statistic is within the *j*-th sub-interval given that ${EWMAT^2}_{t-1}$ is within the *i*-th sub-interval between *0* and *UCL*. To simplify the calculations, we assume that when ${EWMAT^2}_{t-1}$ is in the *i*-th interval, its value coincides exactly with the midpoint of that interval. This approach has been used by many authors [18, 13] and its accuracy has been verified as long as it works with a large number of states. In this case, q_{ij} becomes.

$$q_{ij} = P(L_j < {EWMAT^2}_t < U_j \mid {EWMAT^2}_{t-1} = M_i) \tag{12}$$

Using equation (7), we can express (12) as follows:

$$q_{ij} = P(L_j < r * {T^2}_t + (1-r) * {EWMAT^2}_{t-1} < U_j \mid {EWMAT^2}_{t-1} = M_i)$$
$$q_{ij} = P(L_j < r * {T^2}_t + (1-r) * M_i < U_j)$$

until we finally reach the following expression:

$$q_{ij} = P\left(\frac{L_j - (1-r)M_i}{r} < T^2 < \frac{U_j - (1-r)M_i}{r}\right) \tag{13}$$

The calculation of (13) needs a centralised or non-centralised chi-squared distribution depending on whether the process is in- or out-of-control, respectively.

Let us denote the transition matrix as $\boldsymbol{Q_0}$ when the process is in-control and $\boldsymbol{Q_1}$ when the process is out-of-control. The value of the in-control ARL is obtained using (14):

$$ARL(\boldsymbol{d}=\boldsymbol{0}) = \boldsymbol{v}'(\boldsymbol{I}-\boldsymbol{Q_0})^{-1}\boldsymbol{u} \tag{14}$$

Where $\boldsymbol{u}$ is a vector of ones with the dimension *m x 1*, $\boldsymbol{I}$ is an identity matrix with m x m dimensions, and $\boldsymbol{v}$ is the vector of the probability of initial states with the dimension *m x 1*, and that has all the elements equal to zero except the one that corresponds to the sub-interval that contains the value of $EWMA(T^2)_0$ and is equal to 1.

The zero-state ARL_1 is given by

$$ZARL_1(\boldsymbol{d}=\boldsymbol{d}^*) = \boldsymbol{v}'(\boldsymbol{I}-\boldsymbol{Q_1})^{-1}\boldsymbol{u} \tag{15}$$

and the steady-state ARL_1 is given by

$$SSARL_1 = \mathbf{w}'(\mathbf{I}-\mathbf{Q_1})^{-1}\mathbf{u} \tag{16}$$

where **w**, the m×1, the out-of-control initial state vector is in turn given by

$$\boldsymbol{w} = \frac{\boldsymbol{v}'(\boldsymbol{I}-\boldsymbol{Q_0})^{-1}}{\boldsymbol{ARL}(\boldsymbol{d}=\boldsymbol{0})} \tag{17}$$

2.3 The MEWMA Control Chart

Lowry et *al.* [3] proposed the MEWMA chart, which is an extension of the univariate EWMA. Considering that $\boldsymbol{X} \sim N_p(\boldsymbol{\mu_0}, \boldsymbol{\Sigma})$ and that the vector $\boldsymbol{Z_i}$ is defined as:

$$\boldsymbol{Z_i} = \boldsymbol{r}\boldsymbol{X_i} + (\boldsymbol{1}-\boldsymbol{r})\boldsymbol{Z_{i-1}} \tag{18}$$

The vector $\boldsymbol{Z_i}$ has mean $\boldsymbol{\mu}_0$ when the process is in-control and covariance matrix $\boldsymbol{\Sigma_z} = \frac{\boldsymbol{r}[\boldsymbol{1}-(\boldsymbol{1}-\boldsymbol{r})^{\boldsymbol{2i}}]}{\boldsymbol{2}-\boldsymbol{r}}\boldsymbol{\Sigma}$.

Finally, the statistic for the MEWMA chart is given by:

$$\boldsymbol{T_i^2} = \boldsymbol{Z_i^T}\boldsymbol{\Sigma_z^{-1}}\boldsymbol{Z_i} \tag{19}$$

This chart requires only an upper control limit, UCL, and it is very sensitive to small changes of the in-control mean vector $\boldsymbol{\mu_0}$. These changes are measured in Mahalanobis distance units.

3. OPTIMISATION OF THE EWMAT^2 CONTROL CHART

We used genetic algorithms (GA) to determine the optimal parameters of the proposed control chart. This optimisation technique has been developed using Microsoft Visual Basic Community© and R [19] software to obtain a user-friendly interface that helps to present the different results obtained with this software. For reference, we consider the works by Aparisi and De Luna[20], Epprecht et *al.* [21], and García-Bustos et *al.* [13]. The formal definition of the optimisation problem is:

Input data
In-control ARL: ARL_0
Number of variables p
Mahalanobis distance for which ARL_1 is minimised, $d*$

Objective
Minimise: $SS\text{ARL}(\boldsymbol{d}^*) = SS\text{ARL}_1$
Such that $\text{ARL}(\boldsymbol{d} = \boldsymbol{0}) = \text{ARL}_0$

Decision Variables:
Upper control limit UCL
Smoothing constant r

4. SOFTWARE AND EXAMPLE OF APPLICATION

As mentioned in the previous section, we have developed a program to obtain the optimal parameters of an EWMAT^2 chart. The operation of the program will be explained using the following example of application:

Rojas et *al.* [22] suggested that, in the production of cookies, there are some quality variables in the baking process that need to be monitored: the raw weight before baking (grams), the weight of the baked cookie (grams), volume of the baked cookie (cm^3). Thus, these variables must satisfy the specifications required by the quality departments of the industries that develop these types of processes, because an increase or decrease in the weight or volume of the cookie may cause the packages to be incorrectly sealed and other problems.

It makes sense that these variables have positive correlations because when the volume and weight of the cookies are controlled in the baking process, because the cookies with greater volume obtain greater weight, the weights are found to correspond when measured before and after the baking process. Therefore, for the development of the example of application, we assumed that the correlation matrix of the three aforementioned variables mentioned is:

	Raw W.	Baked W.	Volume
Raw W.	1	0.9	0.7
Baked W.	0.9	1	0.8
Volume	0.7	0.8	1

The objective is to obtain the optimised parameters of the EWMAT^2 control chart to quickly detect the shift vector $\boldsymbol{\mu_1}^{\boldsymbol{T}} = (1,1,1)$ in standard deviation in μ_{01}, μ_{02} , and μ_{03} with the condition that $\text{ARL}_0 = 400$. We have chosen $\text{ARL}_0 = 400$ because on average we want to expect that every 400 samples there will be a false alarm when the process is in control. High ARL values above 200 are commonly used in the literature when the process is under control [13]. It is assumed that the variables are standardised, so the mean vector is $\boldsymbol{\mu_0}^{\boldsymbol{T}} = (0,0,0)$ when the process is in-control. For this example, the Mahalanobis distance is obtained using (2), that is the following operation

$$\sqrt{((1,1,1)-(0,0,0))^T \begin{pmatrix} 1 & 0.9 & 0.7 \\ 0.9 & 1 & 0.8 \\ 0.7 & 0.8 & 1 \end{pmatrix}^{-1} ((1,1,1)-(0,0,0))} = 1.085.$$

For the correct operation of the developed software, the user must consider the following. First, the user must enter the number of variables to monitor and then set the sample size to be analysed; later, it is necessary to introduce the Mahalanobis distance and desired value of in-control ARL (ARL_0). For this example we will assume that the sample size is 1, although other sample sizes are possible.

Finishing the entry of the necessary information, the user presses 'Start' to obtain the following results: The upper control limit, the smoothing constant r, the ARL (d = 0) and steady-state ARL_1. Figure 1 shows the results of the software.

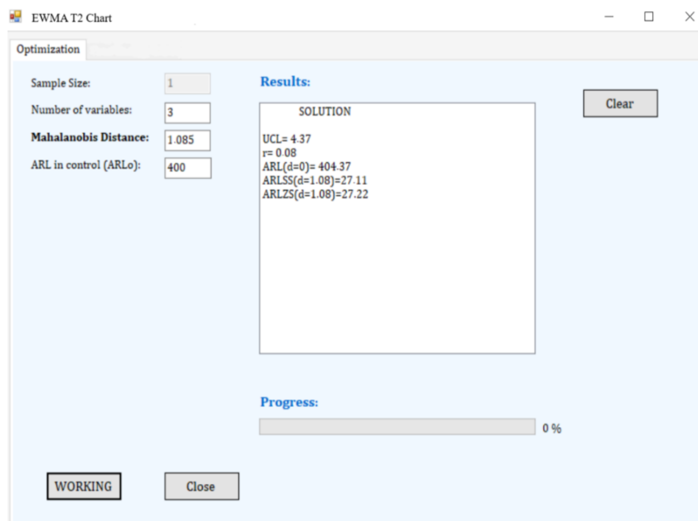

Fig 1: The computer program solving the example of application with the EWMAT^2 control chart.

Considering the results obtained by the software, the optimised EWMAT^2 chart for controlling the cookie baking process uses the statistic $\boldsymbol{EWMAT_i^2} = 0.08T_i^2 + 0.92\boldsymbol{EWMAT_{i-1}^2}$, with $\boldsymbol{EWMAT_0^2} = \boldsymbol{p} = \boldsymbol{3}$, and the upper limit is 4.37. The T^2`statistics must be calculated with values of the standardised variables. When the process is in-control, the EWMAT^2 chart will show a signal in sample number 404 on average, while the change for which the chart has been optimised will be detected in sample number 27 on average.

Although in this section we have used an example of cookie production, the control chart proposed here can be used in any case where the variables involved are normal continuous variables or when the sample size is large and the normal approximation can be used.

To understand how the chart is applied let us consider the following five samples of size one and use the parameters obtained from the software for the implementation of the proposed control chart.

Table 1 presents the sample means, the calculation of the T^2 statistic using the correlation matrix of the example and (4), as well as the calculation of the EWMA T^2 statistic. Figure 2 shows the EWMA T^2 control chart for these data. The last sample presents a signal because its value is greater than UCL.

TABLE 1: SAMPLES FOR EXAMPLE

$\overline{X_1}$	$\overline{X_2}$	$\overline{X_3}$	T^2	EWMA T^2
0.2	0.2	0.2	0.047	0.08*0.047+0.92*3=2.76
0.3	0.2	0.3	0.18	0.08*0.18+0.92*2.76=2.56
1	0.2	0.8	5.05	0.08*5.05+0.92*2.56=2.76
0.5	1.2	1	3.211	2.79
0.2	2.2	0.8	25.19	**4.58**

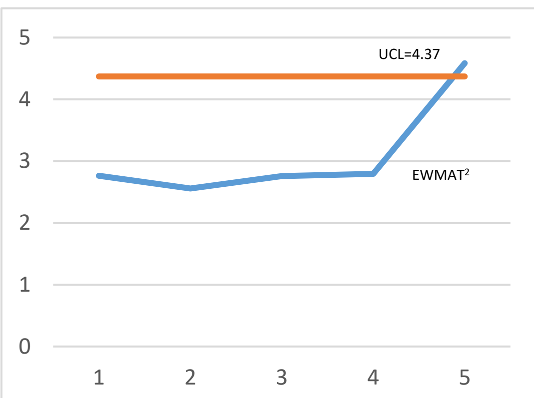

Fig 2: EWMAT^2 control chart for the example

5. COMPARISON OF PERFORMANCE

This section compares the performances between the proposed control EWMAT^2 chart and the Hotelling T^2 and MEWMA control charts. We considered different scenarios, which are presented in Table 2.

TABLE 2: SCENARIOS

Scenario	(ARL_0)	Number of variables
1	200	2
2	200	4
3	200	6
4	200	10

For this analysis, the steady-state ARL_1 is used as a performance measure for EWMAT^2 chart. The ARL_1 values of the T^2 chart were obtained using R[19] software, while the ARL_1 values for the MEWMA chart were taken from Montgomery [23]. In addition, the Mahalanobis distance is used to quantify the change of the in-control mean vector $\mu_{0=}\vec{0}$.

We considered the value of 1 as the sample size, although other larger sizes were evaluated, obtaining results like those presented in this section.

Table 3 presents the comparison between the analysed charts. As we can see, the proposed EWMAT^2 control chart presents a considerable improvement when compared with the Hotelling T^2 chart. This good performance is especially evident in small changes. For example, for a Mahalanobis distance of 0.50 with p =2, the EWMAT^2 control chart has an $SSARL_1$ of 67.38 while the T^2 control chart has an ARL_1 115.53. This means that the proposed control chart, in this case, detects an out-of-control signal in 41.68% fewer steps than the T^2 chart. For a Mahalanobis distance of 0.5 the percentage of improvement is reduced as the number of variables to be analysed increases.

Conversely, when the Mahalanobis distance is greater than 1, the improvement of the EWMAT^2 chart increases as the number of analysed variables increases. For example, when the Mahalanobis distance is 1.5 and the number of variables is 10, the improvement obtained with the EWMAT^2 chart in relation to the T^2 chart is 60.33%, while if p is 2, the improvement percentage is 50%.

As it is optimised for a greater Mahalanobis distance, the smoothing constant r is also found to increase, which indicates that when an optimal EWMAT^2 chart is required to detect large changes, the statistic must give less weight to past observations.

TABLE 3: COMPARISON OF PERFORMANCES, ARL_0=200

Mahalan. Distance	EWMAT^2			T^2	MEWMA		% Difference with T^2	% Difference with MEWMA
	UCL	r	$SSARL_1$	ARL_1	r	ARL_1		
p=2								
0.50	2.52	0.04	67.38	115.53	0.05	26.61	-41.68%	153.20%
1.00	2.85	0.07	18.94	41.92	0.05	11.23	-54.82%	68.63%
1.50	3.23	0.11	7.81	15.78	0.10	6.11	-50.49%	27.82%
2.00	5.81	0.42	4.70	6.88	0.40	3.53	-31.58%	33.25%
3.00	7.01	0.57	1.93	2.16	0.50	1.90	-10.41%	1.80%
p=4								
0.50	4.73	0.04	87.48	138.15	0.05	32.29	-36.68%	170.91%
1.00	4.73	0.04	25.33	60.96	0.05	13.48	-58.45%	87.88%
1.50	6.02	0.14	11.18	24.62	0.10	7.22	-54.57%	54.91%
2.00	6.02	0.14	5.64	10.63	0.10	5.19	-46.97%	8.59%
3.00	7.41	0.27	2.37	2.93	0.30	2.50	-19.11%	-5.16%
p=6								
0.50	7.06	0.05	99.64	149.46	0.05	36.39	-33.34%	173.80%
1.00	7.26	0.06	33.48	74.32	0.05	15.08	-54.95%	122.03%
1.50	7.72	0.09	13.36	32.13	0.10	8.01	-58.43%	66.78%
2.00	7.72	0.09	6.93	14.12	0.10	5.74	-50.92%	20.76%
3.00	8.55	0.15	2.88	3.69	0.20	3.03	-22.00%	-5.11%
p=10								
0.50	11.13	0.04	112.19	161.34	0.05	42.49	-30.47%	164.03%
1.00	12.00	0.08	45.16	92.48	0.10	15.98	-51.17%	182.60%
1.50	12.00	0.08	17.67	44.53	0.10	9.23	-60.33%	91.40%
2.00	12.71	0.12	8.94	20.59	0.10	6.57	-56.57%	36.10%
3.00	13.04	0.14	3.57	5.21	0.10	4.28	-31.38%	-16.48%

In general, we can see that the proposed EWMAT^2 control chart has better performance than the Hotelling T^2 control chart, demonstrating the power of the EWMA charts to detect small shifts. The improvement achieved in this study is similar to that achieved in Epprecht et al. [11], who used an EWMA version to improve the performance of the VDT^2 chart (Variable dimension of T^2). This VDT^2 chart monitors p variables in an adaptive way, depending on the value of statistic ^{T2}, which is calculated with p1< p variables when the value of the statistic is lower than a warning limit, otherwise all the variables must be used to calculate the T^2 statistic.

On the other hand, when the EWMAT^2 chart is compared with the MEWMA chart, the latter continues to present better results.

6. SENSITIVITY ANALYSIS

After confirming the good performance of the proposed control chart EWMAT^2 against that of the Hotelling T^2 control chart, we analyse the sensitivity of the EWMAT^2 control chart. The question posed was: what would happen to the performance of the EWMAT^2 control chart if, with the same optimal parameters of the control chart, the changes that occurred in the process were different? We answered the question using ARL_l; we use four EWMAT^2 control charts optimised for a Mahalanobis distance *d*, and we analyse their performances (ARL_1 values) for different shifts from *d*.

As shown in Figure 3, four cases were proposed for the Mahalanobis distances *d* = 0.5, 0.8, 3, and 3.5; setting *p* = 2 and $ARL_0 = 500$.

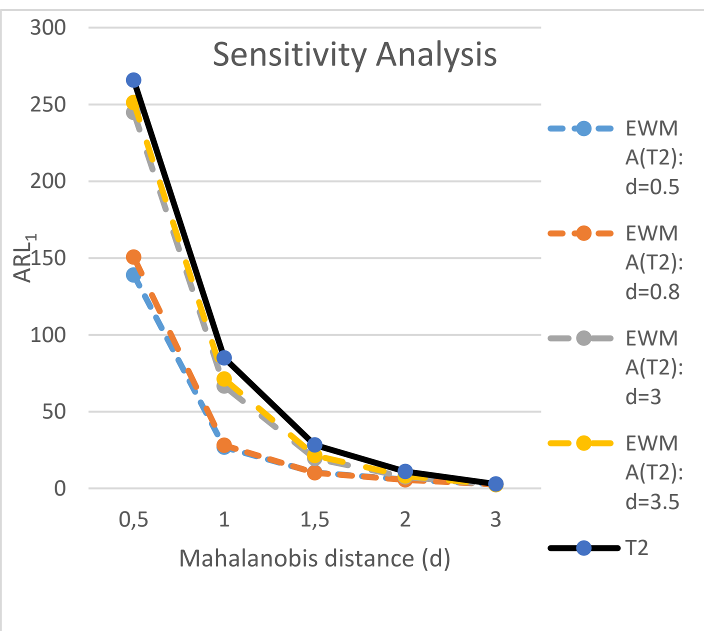

Fig. 3: Sensitivity analysis, EWMAT^2 control chart.

We can observe that all the EWMAT^2 control charts have better performance than the Hotelling T^2 control chart. Similarly, it is observed that the charts with the best yields are those optimised for small changes, as in the case of the chart optimised for *d* = 0.5, which has the best performance.

7. CONCLUSIONS

To improve the performance of the T^2 chart, we propose the EWMAT^2 chart for normal variables. To analyse the performance of the EWMAT^2 chart, we developed a user-friendly program that allows the optimal parameters (UCL, r) to be obtained to quickly detect a change in the vector of means. The performance of the proposed chart is better than that of the T^2 chart for all proposed changes in the analysed cases. This improvement occurs more significantly as the number of controlled variables increases and Mahalanobis distance is greater than 1.

Sensitivity analysis confirmed that the EWMAT^2 chart is sensitive to changes. In general, if the chart is designed and optimised for a Mahalanobis distance $d*$, it will perform well for Mahalanobis distance equal to or greater than $d*$. Therefore, it is recommended that the EWMAT^2 chart is optimised for a small value of Mahalanobis distance.

The proposed chart is an EWMA version of the T^2 chart, while the MEWMA chart is a T^2 version of EWMA charts. When we compare these two charts, the MEWMA chart still performs better.

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