Numerical and Analytical Analysis of a 3UPS-2RPRRR Parallel Robot

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Abstract— This work presents a 3UPS-2RPRRR parallel robot and its geometric analysis. The robot has three rotational degrees of freedom and two translational degrees of freedom in the XZ plane. The inverse geometric model was obtained by closed-loop analysis. The direct geometric model was obtained analytically by the representation of the geometrical varieties for each end limb. The intersection of varieties results in a multivariate nonlinear equation system. The nonlinear system was solved founding four assembly modes. We use the free package SageMath. Two solution methods were addressed: numerical one solution, and Gröbner basis with Toy Buchberger algorithm. Finally, example calculations of inverse and forward geometrical model were analyzed and plotted.

Keywords— parallel robots, inverse kinematics, direct kinematics, nonlinear systems, Gröbner basis, Buchberger algorithm.

1. INTRODUCTION

A parallel robot is a mobile platform attached to a fixed base with several limbs or simple kinematic chains.

Parallel robots are studied because they have advantages like stiffness and speed over serial robots. An important thesis about geometric analysis [1].

Low mobility parallel robots, are known too as constrained mechanisms or sub-six degrees of freedom. These mechanisms have gained special interest because some tasks do not require six degrees of freedom and use less actuators. These kinds of parallel robots are classified by the number and type of degrees of freedom, e.g. rotation or translation.

In the 3UPS-2RPRRR robot, the platform coordinate system can rotate three degrees of freedom and translate two degrees of freedom with respect to a fixed base coordinate system.

The number and type of limbs, with the active joints underlined are represented by several references [2]–[8]. In such representation, the 3UPS-2RPRRR robot has 3 universal-prismatic-spherical and two rotative-prismatic-universal kinematic chains. The prismatic joints are active and the remaining joints are passive.

A literature review about this type of robots reveals that: planar mechanisms, 4 DOF, spherical and Schönflies motions are classified as low mobility parallel robots [9]–[27].

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and mobility of the mechanism. In this work, we use the pitch, roll and yaw convention.

A. Inverse Geometric Analysis

To illustrate the concept, the Fig. 1 shows the schematic in a, and the vector difference in b.

The vectors with endpoints P1, P2, P3, and PC represent the mobile platform. The vectors with endpoints B1, B2, B3, B4, B5 represent the fixed base. The origin O is the start point of all the vectors. The vectors are represented with the corresponding subscript: e.g., \( \vec{B}_1 \) = \( \overrightarrow{OB_1} \), \( \vec{B}_2 \) = \( \overrightarrow{OB_2} \), \( \vec{B}_3 \) = \( \overrightarrow{OB_3} \), \( \vec{B}_4 \) = \( \overrightarrow{OB_4} \), \( \vec{B}_5 \) = \( \overrightarrow{OB_5} \).

The rotation matrix of the platform is:

\[
R = \begin{bmatrix}
    r_{11} & r_{12} & r_{13} \\
    r_{21} & r_{22} & r_{23} \\
    r_{31} & r_{32} & r_{33}
\end{bmatrix}
\]  

(1)

And the position vector at the platform center is:

\[
\vec{P}_c = \begin{bmatrix}
    x_c \\
    y_c \\
    z_c
\end{bmatrix}
\]  

(2)

We use the roll-pitch-yaw convention, also known as Euler ZYX, that is:

\[
R = R_z(\phi)R_y(\theta)R_x(\psi)
\]  

(3)

then, the homogeneous transformation matrix is:

\[
T = \begin{bmatrix}
    c_\phi c_\theta & c_\phi s_\theta s_\psi + s_\phi c_\psi & c_\phi s_\theta c_\psi - s_\phi s_\psi & x_c \\
    s_\phi c_\theta & -s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta c_\psi + c_\phi s_\psi & y_c \\
    0 & 0 & 0 & 1
\end{bmatrix}
\]  

(4)

Here, \( c_\gamma = \cos(\gamma) \) and \( s_\gamma = \sin(\gamma) \) is a compact notation, where \( \gamma \) is \( \psi \), \( \theta \) or \( \phi \).

The points of the base and the platform corresponds to two equilateral circumscribed triangles. The radius \( r_b \) and \( r_p \) are proportional to D:

\[
r_b = \frac{D}{2} \]  

(5)

\[
r_p = \frac{D}{3} \]  

(6)

The vectors at the base are:

\[
\vec{B}_1 = r_b(1,0,0)^T \]  

(7)

\[
\vec{B}_2 = r_b(c_\beta,s_\beta,0)^T \]  

(8)

\[
\vec{B}_3 = r_b(c_-\beta,s_-\beta,0)^T \]  

(9)

\[
\vec{B}_4 = r_b\left(\frac{1}{2},0,0\right)^T \]  

(10)

\[
\vec{B}_5 = r_b\left(\frac{1}{2},0,0\right)^T \]  

(11)

The platform vectors at the base are:

\[
\vec{P}_1^b = r_p(1,0,0)^T \]  

(12)

\[
\vec{P}_2^b = r_p(c_\beta,s_\beta,0)^T \]  

(13)

\[
\vec{P}_3^b = r_p(c_-\beta,s_-\beta,0)^T \]  

(14)

Where:

\[
\beta = \frac{2}{3}\pi \]  

(15)

Applying the transformation to the platform points:

\[
\vec{P}_i = TP_i^b \]  

(16)

For \( i = 1, 2, 3 \), gives the relative position of the platform respect to the base.

The closed-loop vector equations are:

\[
\vec{L}_i = \vec{P}_i - \vec{B}_i \]  

(17)

For \( i = 1, 2, 3 \); and:

\[
\vec{L}_j = \vec{P}_c - \vec{B}_j \]  

(18)

for \( j = 4, 5 \).

Then, the solutions of the square root of the self-dot products are:

\[
q_m = \sqrt{\vec{L}_m \cdot \vec{L}_m} \]  

(19)

Where \( m=1\ldots5 \) and \( q_m \) are the lengths of all the limbs.
B. Forward Geometric Analysis

The forward geometric analysis is often more complex, in this work we follow the geometric approach. The values of the points in the base were defined in section A. All are proportional to D.

The Fig. 2 shows the intersections of the circles C4 with center in B4 and radius q4, and C3 with center in B5 and radius q5 are located in the points PC1 and PC2.

![Fig. 2 Platform central point positions.](image)

Analytically, the central point can be calculated from the intersection between two circles in the XZ plane.

\[(x - B4_x)^2 + (z - B4_z)^2 - q_4^2 = 0 \quad (20)\]

\[(x - B5_x)^2 + (z - B5_z)^2 - q_5^2 = 0 \quad (21)\]

Solving the two-equation system, we got two points PC1 and PC2. The points are symmetrical to the XY plane, we choose the point with positive z. Renaming as point PC = (x_c, 0, z_c), with coordinates:

\[x_c = \frac{q_4^2 - q_5^2}{D} \quad (22)\]

\[z_c = \frac{-16q_4^4 + 8D^2q_5^2 - 16q_4^2 + 8(D^2 + 4q_5^2)q_4^2}{4D} \quad (23)\]

The Fig. 3 shows the intersection of two spheres. The central sphere with center in PC and radius r_p, and the sphere with center in B1 and radius q1.

![Fig. 3 Intersection of two spheres.](image)

For each intersection between the spheres with center in B1, B2, B3 and radius q1, q2, and q3, respectively and the central sphere with center in PC and radius r_p. The equations are defined by:

\[(x_i - B_i)^2 + (y_i - B_i)^2 + (z_i - z_c)^2 - r_p^2 = 0\]

For i = 1, 2 and 3. The points Bi = (B_i x, B_i y, B_i z) are the vertexes of the base triangle. The points Pi = (x_i, y_i, z_i) are the vertexes of the platform, and PC = (x_c, y_c, z_c) is the center point of the platform.

The difference between these two groups of equations gives:

\[\frac{D^2 - 2q_4^2 + 2q_5^2}{D} \cdot x_1 + \frac{61}{144}D^2 - q_4^2 - \frac{1}{2}q_4^2 - \frac{1}{2}q_5^2 = 0 \quad (26)\]

\[\frac{D^2 + 4q_4^2 - 4q_5^2}{2D} \cdot x_2 - \frac{1}{2}\sqrt{3}D \cdot y_2 - \frac{\gamma}{2D} \cdot z_2 + \frac{61}{144}D^2 - q_4^2 - \frac{1}{2}q_4^2 - \frac{1}{2}q_5^2 = 0 \quad (27)\]

\[\frac{D^2 + 4q_4^2 - 4q_5^2}{2D} \cdot x_3 + \frac{1}{2}\sqrt{3}D \cdot y_3 - \frac{\gamma}{2D} \cdot z_3 + \frac{61}{144}D^2 - q_4^2 - \frac{1}{2}q_4^2 - \frac{1}{2}q_5^2 = 0 \quad (28)\]

Where:

\[\gamma = \sqrt{-D^4 + 8D^2q_4^2 - 16q_4^4 - 16q_5^4 + 8(D^2 + 4q_5^2)q_4^2} \quad (29)\]

The coordinates of the points Pi = (x_i, y_i, z_i) for i = 1, 2, and 3, are the nine unknowns.

The center of an equilateral triangle is PC, then the average of the three points is the center of the platform:

\[P1 + P2 + P3 = 3 \cdot PC \quad (30)\]

This property gives three more equations:

\[x_1 + x_2 + x_3 - 3 \cdot x_c = 0 \quad (31)\]

\[y_1 + y_2 + y_3 - 3 \cdot y_c = 0 \quad (32)\]

\[z_1 + z_2 + z_3 - 3 \cdot z_c = 0 \quad (33)\]

The Fig. 4 shows the distances between the platform points P1, P2, P3 and PC.

![Fig. 4. Distances between platform points.](image)
The sides of the equilateral triangle of the platform are equal to:

\[ s_{12} = s_{23} = s_{31} = s \]  

(34)

By cosines law:

\[ s^2 = 2r^2 - 2r^2 \cos s \left( \frac{2}{3} \pi \right) = 2r^2 \left( 1 + \frac{1}{2} \right) = 3r^2 \]  

(35)

The relation between the distance s and the points coordinates are:

\[ (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - s^2 = 0 \]  

(36)

\[ (x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2 - s^2 = 0 \]  

(37)

\[ (x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2 - s^2 = 0 \]  

(38)


III. MATERIALS AND TOOLS

In this work we solve the inverse and direct geometric methods by using SageMath 8.5 [29] installed in Windows 10. Running in an Intel core i7 at 2.5Mhz with 16Gb RAM.

The SageMath scripts and results are showed in the Appendix. The scripts are implementation of the geometrical analysis.

The solution methods applied are part of the libraries in SageMath.

IV. RESULTS

In this section, we use the methods of section III with example values. The base and platform dimensions are proportional to D, we use a scale base of D = 1. This means that the robot can be scaled to any size proportional to D.

The x values are in the interval [-D/2, D/2], the y value is zero, and z values are in the interval [-D/4, D/4]. The orientation angles \( \psi, \theta, \) and \( \phi \) are in the interval (-\( \pi/3, \pi/3 \)).

Solving the inverse geometric is easy and with a different position and orientation values of the Table I.

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>( z_i )</th>
<th>( \psi )</th>
<th>( \theta )</th>
<th>( \phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1/2</td>
<td>1/4</td>
<td>-( \pi/3 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-1/4</td>
<td>3/8</td>
<td>-( \pi/4 )</td>
<td>( \pi/6 )</td>
<td>( \pi/6 )</td>
</tr>
<tr>
<td>0</td>
<td>1/2</td>
<td>-( \pi/6 )</td>
<td>( \pi/4 )</td>
<td>( \pi/4 )</td>
</tr>
<tr>
<td>1/4</td>
<td>5/8</td>
<td>0</td>
<td>-( \pi/3 )</td>
<td>-( \pi/3 )</td>
</tr>
<tr>
<td>3/8</td>
<td>3/4</td>
<td>( \pi/6 )</td>
<td>-( \pi/3 )</td>
<td>-( \pi/3 )</td>
</tr>
<tr>
<td>5/8</td>
<td>7/8</td>
<td>( \pi/4 )</td>
<td>-( \pi/4 )</td>
<td>-( \pi/4 )</td>
</tr>
<tr>
<td>1/2</td>
<td>1</td>
<td>( \pi/3 )</td>
<td>-( \pi/6 )</td>
<td>-( \pi/6 )</td>
</tr>
</tbody>
</table>

The values of the output for each pose are in the TABLE II.

<table>
<thead>
<tr>
<th>( q_1 )</th>
<th>( q_2 )</th>
<th>( q_3 )</th>
<th>( q_4 )</th>
<th>( q_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.712</td>
<td>0.506</td>
<td>0.712</td>
<td>0.353</td>
<td>0.790</td>
</tr>
<tr>
<td>0.560</td>
<td>0.568</td>
<td>0.68</td>
<td>0.375</td>
<td>0.625</td>
</tr>
<tr>
<td>0.456</td>
<td>0.665</td>
<td>0.866</td>
<td>0.559</td>
<td>0.559</td>
</tr>
<tr>
<td>0.402</td>
<td>0.874</td>
<td>1.067</td>
<td>0.800</td>
<td>0.625</td>
</tr>
</tbody>
</table>

The platform points for \( xc = -\frac{D}{2}, zc = \frac{D}{4}, ps = -\frac{D}{4} \) and \( th = ph = 0 \) are in the Table III.

<table>
<thead>
<tr>
<th>Point</th>
<th>( X )</th>
<th>( y )</th>
<th>( z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>-0.166</td>
<td>0.0</td>
<td>0.25</td>
</tr>
<tr>
<td>P2</td>
<td>-0.666</td>
<td>0.1447</td>
<td>0.0</td>
</tr>
<tr>
<td>P3</td>
<td>-0.666</td>
<td>-0.144</td>
<td>0.5</td>
</tr>
</tbody>
</table>

The values of \( q_0 \) are in the first file at the Table II. The computations of the forward geometrical model found the solution by numerical approximation. Eight roots were found, six complex and two reals in the Table IV and Table V.

<table>
<thead>
<tr>
<th>Point</th>
<th>( X )</th>
<th>( y )</th>
<th>( z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>-0.194</td>
<td>-0.073</td>
<td>0.138</td>
</tr>
<tr>
<td>P2</td>
<td>-0.696</td>
<td>-0.205</td>
<td>0.075</td>
</tr>
<tr>
<td>P3</td>
<td>-0.608</td>
<td>-0.131</td>
<td>0.536</td>
</tr>
</tbody>
</table>

The Toy Buchberger algorithm and polynomial root found one solution (See the Table VI). This solution is coincident with the first numerical real solution.

<table>
<thead>
<tr>
<th>Point</th>
<th>( X )</th>
<th>( y )</th>
<th>( z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>-0.194</td>
<td>-0.073</td>
<td>0.138</td>
</tr>
<tr>
<td>P2</td>
<td>-0.696</td>
<td>-0.205</td>
<td>0.075</td>
</tr>
<tr>
<td>P3</td>
<td>-0.608</td>
<td>-0.131</td>
<td>0.536</td>
</tr>
</tbody>
</table>

The Fig. 5 shows the plot of the inverse geometric problem with the initial data, when the direct numerical method diverges.

![Fig. 5 Inverse geometric plot](image-url)
When the direct numerical method does not diverge, the initial conditions must be closest to the solution searched.

For example, in the Fig. 7 an inverse solution is found by direct numerical method and Gröbner basis method.

The two corresponding z positive solutions, in red and green are showed in the Fig. 8.

Another two solutions, corresponding to the z negative are showed in the Fig. 9.

**CONCLUSIONS AND RECOMMENDATIONS**

The inverse geometric analysis of the 3UPS-2RPRR robot was solved by using the transformation matrix, no working modes were found because the limbs are simple prismatic joints.

SageMath is a free open tool for solving nonlinear systems and algebraic manipulation.

The accuracy solution of the forward geometric analysis can be found by using Gröbner basis and Toy Buchberger algorithm.

The 3UPS-2RPRR parallel robot has four assembly modes. Near to the origin the Newton Raphson method can oscillate between close values.

Sometimes the limbs cross and has not physical meaning. The analysis was made for all the possible values in the space of the numerical solution.

Is recommended to avoid the poses when the position oscillates. This issue must be addressed in the control of the robot. The yaw may be in positive values far from zero.

The execution time for inverse analysis must be implemented in real time because is important in motion planning.

The 3UPS-2RPRR robot can be applied in simulators of forward and backward displacements like cars, boats or spaceships.

**APPENDIX**

The scripts in SageMath.

A. **Inverse Geometric Model**

The script for initial values is:

```plaintext
p=2
rb=D/2
rp=D/3
xc=D/4
yc=0
zc=D/3
ps=pi/12
th=0
ph=0
```
\begin{align*}
B1 &= \text{vector([rb, 0, 0])} \\
B2 &= \text{vector([rb*cos(2/3*pi), rb*sin(2/3*pi), 0])} \\
B3 &= \text{vector([rb*cos(-2/3*pi), rb*sin(-2/3*pi), 0])} \\
B4 &= \text{vector([-rb/2,0,0])} \\
PB1 &= \text{vector([rb,0,0])} \\
PB2 &= \text{vector([rp*cos(2/3*pi), rp*sin(2/3*pi), 0])} \\
PB3 &= \text{vector([rp*cos(-2/3*pi), rp*sin(-2/3*pi), 0])} \\
P &= \text{vector([sx, sy, sz])}
\end{align*}

The script defining the transformation matrix, the points and the differences is:

\texttt{sage: R=matrix([[sx, nx, ax], [sy, ny, ay], [sz, nz, az]])}
\texttt{sage: nz=cos(th)*sin(ps)}
\texttt{sage: sx=cos(ph)*cos(th)}
\texttt{sage: eqp}
\texttt{sage: eqp}
\texttt{sage: lb1=line([P1,B1])}
\texttt{bas=polygon([B1, B2, B3], color='green')}
\texttt{sage: n(vector([q1,q2,q3,q4,q5]))}
\texttt{sage: q2=sqrt(L2.dot_product(L2))}
\texttt{sage: q1=sqrt(L1.dot_product(L1))}
\texttt{sage: q3=sqrt(L3.dot_product(L3))}
\texttt{sage: q4=sqrt(L4.dot_product(L4))}
\texttt{sage: q5=sqrt(L5.dot_product(L5))}
\texttt{n(vector([q1,q2,q3,q4,q5]))}
\texttt{p1=polyline([P1, P2, P3], color='red')}
\texttt{bas=polyline([B1, B2, B3], color='green')}
\texttt{lb1=line([P1,B1])}
\texttt{lb2=line([P2,B2])}
\texttt{lb3=line([P3,B3])}
\texttt{lb4=line([PC,B4])}
\texttt{lb5=line([PC,B5])}
\texttt{crb=circle((S[0],0),rb)}
\texttt{show(crb+p1+p2+p3+p4+p5)}

The script for calculation and plotting the schematic is:

\texttt{q1=sqrt(L1.dot_product(L1))}
\texttt{q2=sqrt(L2.dot_product(L2))}
\texttt{q3=sqrt(L3.dot_product(L3))}
\texttt{q4=sqrt(L4.dot_product(L4))}
\texttt{q5=sqrt(L5.dot_product(L5))}
\texttt{n(vector([q1,q2,q3,q4,q5]))}
\texttt{p1=polyline([P1, P2, P3], color='red')}
\texttt{bas=polyline([B1, B2, B3], color='green')}
\texttt{lb1=line([P1,B1])}
\texttt{lb2=line([P2,B2])}
\texttt{lb3=line([P3,B3])}
\texttt{lb4=line([PC,B4])}
\texttt{lb5=line([PC,B5])}
\texttt{crb=circle((S[0],0),rb)}
\texttt{show(crb+p1+p2+p3+p4+p5)}

The resulting equations are in terms of \(x_1, y_2, z_3\).

The identifiers of the equations 36-38 are eqp1, eqp2, and eqp3. Substituting the solutions in equations eqp11, eqp12, eqp31, respectively:

\begin{align*}
\text{eqp11} &= \text{eqp1.substitute(x2s,x3s,y1s,y3s,z1s,z2s)} \\
\text{eqp12} &= \text{eqp2.substitute(x2s,x3s,y1s,y3s,z1s,z2s)} \\
\text{eqp31} &= \text{eqp3.substitute(x2s,x3s,y1s,y3s,z1s,z2s)}
\end{align*}

We got three nonlinear equations in terms of \(x_1, y_2, z_3\).

Two solution methods in SageMath were applied: Numerical and the Toy Buchberger algorithm for Gröbner basis.

The numerical method is simple, but sometimes the solution diverges.

\texttt{solt=solve([eqp11,eqp12,eqp31],x1,y2,z3)}

Using the Gröbner basis method, we take the left side expression and redefine the variables as a polynomial in \(x, y, z\):  

\texttt{R, (x, y, z) = PolynomialRing(RationalField(),3, 'xyz').objgens()}
\texttt{eqp11=eqp11.subs(x=x1,y=x2,z=x3)}
\texttt{eqp12=eqp12.subs(x=x1,y=x2,z=x3)}
\texttt{eqp31=eqp31.subs(x=x1,y=x2,z=x3)}

Then, define the ideals with Gröbner fan function, we got three functions:

\texttt{I = (eqp11s, eqp21s, eqp31s)*R1}
\texttt{g = I.groebner_fan()}  
\texttt{F=g.buchberger()}  
\texttt{fun0=F[0]}  
\texttt{fun1=F[1]}  
\texttt{fun2=F[2]}  

The first function is in \(z\) variable terms, we redefine as a polynomial in a \(zu\) variable and numerically find the roots. Finally, we replace in the two solutions, in terms of \(xv, y\):

\begin{align*}
\text{R.<zu> = PolynomialRing(QQ)}  
\text{fun0zu=fun0.subs(z=zu)}  
\text{soltz1=find_root(fun0zu,0,z_c+D)}
\end{align*}
solgy1=n(-fun1.subs(z=solgz1)+y)
solgx1=n(-fun2.subs(z=solgz1)+x)

Then, we got the values of $x_1, y_2$ and $z_3$. Finally, the system is solved by substitution of these values in the equations 39-44.

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