The use of the Partial Differential Equations for Application in Inpainting

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***Abstract*– This paper presents the use of the Partial Differential Equations (PDE) for application in Inpainting. Also, it is described basic concepts of the image processing leading to restauration. Thus, inpainting is interesting and powerful method of restauration, which the last decades have been studied by several researchers. Finally, it is important take into account that this topic was chosen due to include of PDE, discretization and matrices.**

***Keywords--* Image Processing, Inpainting, Partial Differential Equations, Non-linear Equation, and Anisotropic Diffusion.**

I. Introduction

Image inpainting, also known as image interpolation, is a technique used for recovering a masked image by using the information of the visible part of the image. The goal of this technique is that the modifications are done in a way that is undetectable for an observer who does not know the original image. It's been widely used to remove an object in an image, filling patches in an image, etc. Since this process allows eliminate noise and improve the brightness, color and detail of an image. Thus, it is used in various image restoration applications, such as the reconstruction of missing or damaged parts of a piece of art, or for the recovery of lost blocks in the encoding and transmission of images.

The term inpainting was invented by art restoration workers [1, 2] and first appeared in the framework of digital restoration in the work of [1, 3] (PDE approach).

(a) (b)

Fig. 1 An example of image inpainting used in art restoration.

(a) Original Image, and (b) In-painted. [Sharkey, 1996].

Figure 2 considered the special case of the noisy step edge. It is noticed the gaussian smoothing across the edge, which makes a sense away from the edges. However, to preserve the edges is needed to do something different around the edge. Thus, assuming that smoothing is not the same everywhere (i.e. whole image), this result is known as non-homogenous. On another hand, assuming that so smoothing is influenced by local image properties, i.e. the smoothing depends on the degraded image, and not the original image. As result, it eliminates some unimportant structure, which should not continue to affect the smoothing. This operation is known as diffused image, and as it gets from a non-linear equation is called non-linear diffusion [4, 5].

The direction of the flux depends on image properties, if the intensities are linear (i.e. form ramp). For instance, the flux to be in the direction in which the ramp is going down, which is the direction of the gradient. This is a one-dimension problem, with the same direction. If the flux is always in the direction of the gradient, it is called the isotropic diffusion. However, near step edge, the flux to be parallel to the edge, which is perpendicular to the gradient. Thus, if the direction of the flux varies in the image, it is called anisotropic.

According to the previous information of the image inpainting and its applications, this paper presents the following topics:

1) Explain the fundamentals of image inpainting technique (mathematical basics: Partial Differential Equations - PDE, calculus of variations);

2) Explain about some methods on image inpainting (the research progress in image inpainting); and

3) Describe the results of the implementation.

II. Basic Concepts of Restoring an Image - Inpainting

A. Restoration of an Image

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An image can be expressed as a function binomial f, that from R^2 to R, where $f(x, y)$ is the intensity of the image at position (x, y). Also, a color image is represented by three functions, which are a vector value function:

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$$f:[a,b]\times[c,d]\rightarrow[0,1] \tag{1}$$

$$f(x,y)=\begin{bmatrix} r(x,y) \\ g(x,y) \\ b(x,y) \end{bmatrix} \tag{2}$$

Figure 2 shows a general scheme of the degradation and restoration of an image. Degradation is commonly due to the factors, such as degradation filter and noise. Thus, to recover or estimate the image through of the observed image is performed a stage known as restoration.

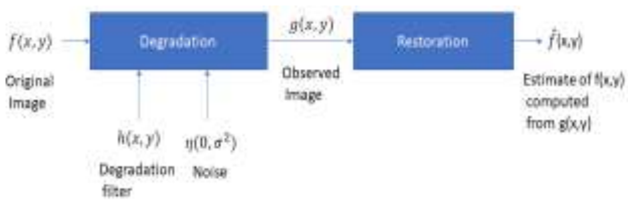


Fig. 2 General scheme of the degradation and restoration of an image.

There are many methods to solve the problem of the degradation of an image, such as the elimination of noise, among others [6]. In essence, it is important to make the restoration but at the same time preserve the structure of the image (i.e. losing information of the image). A method usual to restorer an image is the linear filtering, this choice is used mainly due to its computational cost advantage. The disadvantage of linear filtering is that it has no feedback, so the filtering does not detect contours and the processing is done in the same way throughout the image [6, 7]. The importance of preserving the contours in an image is established in other methods of inpainting, which in the last two decades has been established mainly in the use of mathematics, i.e., Partial Differential Equations (PDE) [8].

III. Heat Equation and Non-Linear Diffusion

In inpainting, the applications using PDE are based on with the diffusion equation [9]. Considering the analogy between the scalar field of temperatures in a diffusion process and the scalar field of the gray level of an image in a smoothing filtering, the dynamics of both processes respond to the expression:

$$\partial_t I = div(D\cdot\nabla I) \tag{3}$$

On another hand, diffusion can be described as a physical process that equilibrates temperature differences without creating or destroying mass. It can be expressed by Fick's law:

$$j=-D\cdot\nabla I \tag{4}$$

Equation 4 represents the equilibration property, where is a concentration gradient ∇I and is its flux j. D is a diffusion tensor, which describes a relation between ∇I and j. Also, if ∇I and j are parallel then it is called isotropic, in otherwise it is called anisotropic [9, 10]. Since the diffusion does only transport mass complying the equilibration property, then by the continuity equation:

$$\partial_t I=-div(j) \tag{5}$$

Thus, the diffusion equation (i.e. in Eq. 6) is obtained from the equations 4 and 5,

$$\partial_t I=div(D\cdot\nabla I) \tag{6}$$

Additionally, it is observed that heat equation is expressed as:

$$\partial_t I=\frac{\partial I}{\partial t}=\Delta I=\frac{\partial^2 I}{\partial x^2}+\frac{\partial^2 I}{\partial y^2} \tag{7}$$

Heat equation in divergence form:

$$\partial_t I=\frac{\partial I}{\partial t}=\Delta I=div(\nabla I) \tag{8}$$

Comparing the equation 6 and 8, if diffusion tensor D is equal to 1 then both equations are equal. In image processing the heat equation is very useful and it is known as linear diffusion filtering. However, linear diffusion filtering has some limitations to take into account such as: (1) smooth noise and blur import features, (2) dislocate edges, and (3) some smoothing properties do not apply to 2-D. Figure 3 shows the image reconstructed using the heat equation, which works well only when the areas are homogenous and in otherwise it's causing the edge to be lost in the restored in sharp edge, the edge pixels are diffused.

Figure 3. (a) An image of an edge with a hole, and (b) the same image reconstructed using the heat equation.

A. *Non-Linear Diffusion*

Considering the following diffusion equation:

$$\frac{\partial I}{\partial t}=div(c(x,y,t)\nabla I) \tag{9}$$

Where $c(x,y,t)$ is the diffusion coefficient (or conduction coefficient). This is isotropic diffusion when $c(x,y,t)$ is a constant. Let $E(x,y,t)$ be the estimate of edges, then the diffusion coefficient could be:

$$c(x,y,t)=g(\|E\|) \tag{10}$$

where $g(\bullet)$ is some scalar function of $\|E\|$. In other words, the diffusion tensor D is $g(\|E\|)$. If $g(\|E\|)$ only depends on the initial image, then it is effectively constant and, so it gets a linear PDE. Equation 10 described a function that decreases with $\|\nabla f\|$, an option to get a non-linear diffusion,

$$g(\|\nabla f\|^2) = \frac{1}{\sqrt{1 + \frac{\|\nabla f\|^2}{\lambda^2}}} \tag{11}$$

Notice this decreases from 1 to 0 as $\|\nabla f\|$ grows. When $\|\nabla f\| = \lambda;\ g(\|\nabla f\|^2) = 1/\sqrt{2}$.

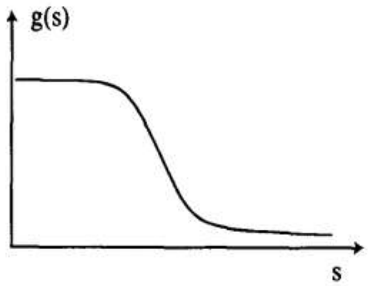


Fig. 4 The quality shape of nonlinearity $g(s)$

B. Perona and Malki

Furthermore, $g(\cdot)$ is going to depend on the current image, partially smoothing image, not the initial image. A good choice of $\|E\|$ would be:

$$E(x, y, t) = \nabla I(x, y, t) \tag{12}$$

Perona-Malik then introduce a nonlinear extension of it:

$$\frac{\partial I}{\partial t} = div(g(\|\nabla I\|)\nabla I) \tag{13}$$

To limit diffusion around edges, two functions have been proposed:

$$g(\|\nabla I\|) = e^{-\left(\frac{\|I\|}{\alpha}\right)^2} \tag{14}$$

$$g(\|\nabla I\|) = \frac{1}{1 + \left(\frac{\|\nabla I\|}{\alpha}\right)^2} \tag{15}$$

IV. Isotropic and Anisotropic Diffusion

As previously seen, an isotropic diffusion characteristic that the edges are not respected and the link between the regions with destroyed. Anisotropic diffusion is used to solve this type of problem this operator can preserve the position of the edges and homogenize (or blur) other areas. Generalization and conservation of the edges through the heat equation, however the non-homogeneous diffusion coefficients are considered:

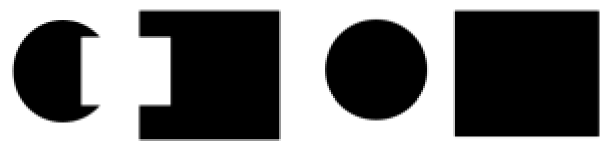
$$\frac{\partial I}{\partial t} = \nabla \cdot (c(x, y, t)\nabla I) = c(t, x, y)\Delta I + \nabla c \cdot \nabla I \tag{16}$$

V. Implementation in Matlab and Results

What's Needed to Inpaint? the image to be restore need the followings stages:

1) the mask image (i.e. to delimit the region of the image to be in-painted), and

2) the restoration loop: to perform m steps of inpainting, n steps of diffusion, m steps of inpainting and so on.



(a) (b)

Fig. 5 (a) Image is going to need to be inpainted; (b) Original image (i.e. our natural guess)

The inpainting problem can be state as: given Ω, which stand for the region to be inpainted, and ∂Ω for its boundary. Thus, to make sure that: (1) the structure of the area surrounding Ω is continued, and (2) The color in continued inside Ω.

A. Description of the Inpainting Algorithm - PDE

Let $I_0(i, j)$ be a discrete 2D gray level image. Form a family of images $I(i, j, n)$ such that $I(i, j, 0) = I_0(i, j)$ and $\lim_{n \to \infty} I(i, j, n) = I_R(i, j)$, where I_R is the inpainted image. Generally, the inpainting algorithm can be written as:

$$I^{n+1}(i, j) = I^n(i, j) + \Delta t I_t^n(i, j), \forall (i, j) \in \Omega \tag{17}$$

Where n denotes the inpainting 'time'. $I^{n+1}(i, j)$ is an improved version of $I^n(i, j)$, $I_t^n(i, j)$ represents the enhancement of the image, this evolution equation only runs inside Ω, the region to be inpainted. How to model $I_t^n(i, j)$? And then it is described the inpainting algorithm:

Taken into account that the goal is to smoothly propagate information from outside Ω into Ω.

- $L^n(i, j)$ – the information that we want to propagate
- $\vec{N}^n(i, j)$ – the propagation direction
- $I_t^n(i, j) = \overrightarrow{\delta L^n}(i, j) \bullet \vec{N}^n(i, j)$ (also written as $\nabla L \bullet \vec{N}$ for continuous mathematics)

Where $\overrightarrow{\delta L^n}(i, j)$ is a measure of the change in the information $L^n(i, j)$. At steady state, $I^{n+1}(i, j) = I^n(i, j)$, i.e.,

$\overrightarrow{\delta L^n}(i,j) \bullet \vec{N}^n(i,j) = 0$, meaning that the information has been propagated in the direction $\vec{N}$.

- $L^n(i,j)$ – smoothness estimator → Laplacian: $L^n(i,j) = I^n_{xx}(i,j) + I^n_{yy}(i,j)$
- $\vec{N}^n(i,j)$ – the time-varying isophotes directions $\nabla^{\perp} I^n(x,y)$, ($\nabla^{\perp} = (\partial_y, \partial_x)$, the direction of the smallest change. Isophotes are the level lines of equal gray levels.) - i.e., $L = \Delta I, \vec{N} = \nabla^{\perp} I$. Substitute L, $\vec{N}$ into the equation $\nabla L \bullet \vec{N} = 0$, it is:

$$\nabla(\Delta I) \bullet \nabla^{\perp} I = 0 \tag{18}$$

Figure 6 shows (a) the isophotes, (b) the propagation direction to be the normal to the boundary of the region to be inpainted Ω, and finally (c) image to be inpainted, and (d) inpainted image output.

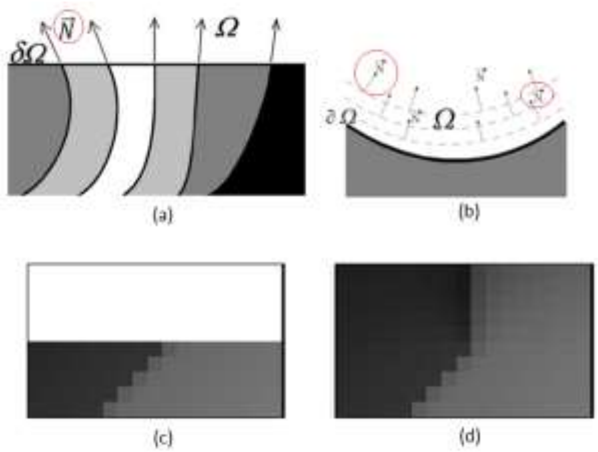


Fig. 6 (a) The isophotes, (b)propagation direction as the normal to signed distance to the boundary of the region inpainted Ω, (c) image to be inpainted, and (d) inpainted image output.

B. Results

Figure 7 and 8 shows as the PDE method depend of the number of iterations. Also, the restoration image enhances with the increasing of the number of iterations, and it preserves information in the edges. The smoothing is almost imperceptible in comparison with the result observed in the isotropic diffusion model. Additionally, the main limitation of this method is that the structure (geometry) is restored but the texture is not.

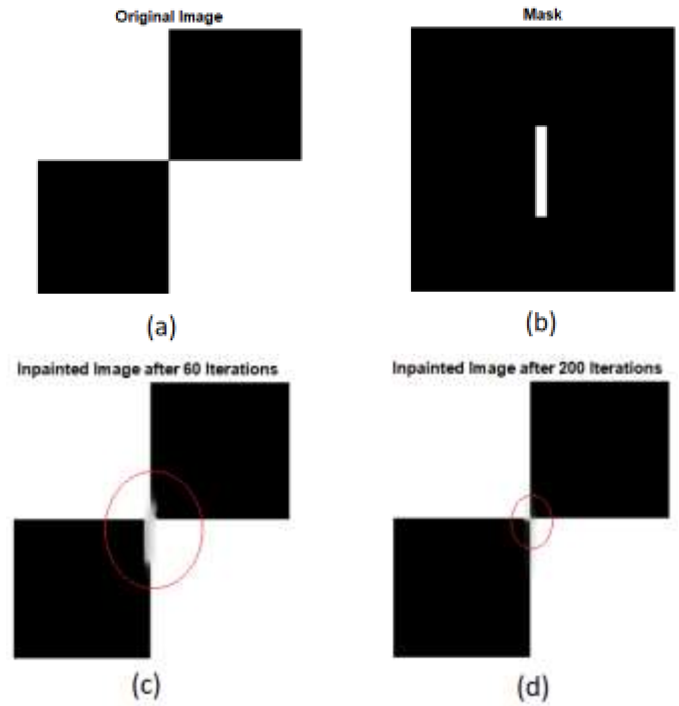


Fig. 7 Results of the algorithm test implemented in MATLAB, in a synthetic image: (a) Original image, (b) mask of the image, (c) and (d) are the output image for 60 iterations and 200 iterations.

VI. Conclusions

- The theoretical framework of linear, nonlinear, isotropic and anisotropic diffusion has been presented, paying special attention to the diffusion of nonlinear and anisotropic PDE.
- PDE is a very useful and versatile tool in image restoration applications.
- It was corroborated the great advantage of working with PDE, since it allows to extract information from the image that other methods do not allow.
- It is only necessary to establish the initial conditions so that the PDE can automatically calculate the value of the missing intensities and restoring the observed image.
- The implementation of the algorithm is interesting because a discretization of different models of equations is performed.

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