Intuitive explanations of basic engineering-related mathematical concepts

***Abstract** - Current students learn differently: many instructors observe less textbook-reliance and more dependence on web-based explanations, short videos, animations, and demonstrations. When it comes to concept comprehension, students repeatedly miss the Aha! Moment, and ask for more hands-on, experiential, visual, intuitive, fun (e.g., game-based), and tech-based information. Clearly, basic concepts should be introduced in easy-to-comprehend, visual, and intuitive ways. This is most relevant in math courses that are usually taught with little or no connection to other disciplines, and in particular engineering. This paper focuses on introducing basic math concepts by linking them to daily experiences using relevant analogy-based examples, to be introduced prior to delving into purely mathematical explanations and proofs. The paper uses tangible examples for visualizing some concepts in algebra and set theory, as well as for visual interpretation of large and small numbers.*

Keywords-Visual, intuitive, learning, math, algebra, Venn.

1 INTRODUCTION

This paper introduces some ideas for explaining engineering-related mathematical concepts by linking them to daily experiences. The focus is on visual and intuitive experience-based explanations of some math concepts. The examples are meant to provide additional material for introductory purposes only, to allow students to see the relevance of math to their daily life. They are intended to allow learners to not only recognize and appreciate the relevance of math to everyday life, but also tap on different learning styles and keep learners engaged, thereby allowing for multiple and diverse ways of comprehension.

It is important to emphasize that the material presented in this paper is meant to be add-ons to existing calculus textbooks, and that it is not meant to suggest competition, modifications or replacement of existing textbooks.

The material is referred to as work in progress and is to be shared and discussed with multiple audiences. When these and many other examples were used, students have demonstrated better, clearer understanding of difficult concepts, and praised the approach. Even though this was not an official assessment, based on similar experience that was gained and assessed by the author in other engineering and science related subjects (Control Systems, Digital Signal Processing, Computer Algorithms, and Physics), it is believed that the approach has a great potential. Students not only have commended the approach, but they have demonstrated its effectiveness.

The rational for this work stems from observations that the current generation of students learn differently: less textbook-reliance, and more dependence on web-based explanations, such as short videos, animations, and demonstrations. When it comes to concept comprehension, students repeatedly miss the “Aha! Moment,” and ask for more hands-on, experiential, visual, intuitive, fun (e.g., game-based), tech-based, and web-based information.

This is not new. For example, Tyler DeWitt [1] recognized this problem and taught isotopes to high school students using analogy to same cars but with minor changes to illustrate that isotopes are basically the same atom, i.e., have the same number of protons and electrons with varying number of neutrons. By focusing on calculus there are some books that include visual explanations (see for example references [2-10]). Of a special interest is the work by Apostol and Mamikon from Caltech [11,12]. They were able to explain integration of some functions without using mathematical formulas. The author of this paper published papers on this topic [13-20] in addition to books [21,22], one for understanding concepts in “Control Systems” and the other for

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understanding the basics of "Newton's Laws of Motion."

The bigger picture

This work is part of a multi-modal integrated project aimed at understanding concepts in STEM. The approach is meant to help both teachers and students, thereby allowing for more innovative teaching and comprehension-based learning. The project is catered towards appealing to learners in visual, intuitive, and interactive/engaging means. It uses daily-life and relevant experiences, as well as different STEM/ STEAM examples and activities. The project targets a broad understanding and appreciation of basic concepts in STEM, currently involving Physics/Mechanics, Calculus, Statics, Control Systems, Digital Signal Processing (DSP), Probability, Estimation, and Computer Algorithms. Though the material can be used by teachers and learners in classroom settings, it is primarily designed to (eventually) be web-based, targeting those who prefer self-paced self-learning friendly environments. Simply put, the project is principally designed for a learner-centered e-based environment, making it ready for large scale dissemination. Examples of calculus concepts that the author and his team plan to develop and integrate include: (a) games, (b) puzzles and teasers, (c) animations, (d) visual and intuitive daily-experiences-based examples, (e) movies and short video clips, (f) demonstrations, (g) hands-on activities (including those based on virtual reality and augmented reality), (h) teaming and communication exercises, (i) small-scale inquiry-based research, (j) presentations, and peer-based teaching/learning, (k) visual click-based e-book, (l) community and social engagement, and (m) challenges beyond the basics.

2 EXAMPLES

The following is a set of examples for visualizing ideas in algebra and set theory, as well as for visual interpretation of large and small numbers.

Algebra Examples

2.1 Visualizing the meaning of "function": meat processor example

Figure 1 shows a mechanical meat processor where the input (IN) is raw meat and the output (OUT) is its processed version. Clearly one cannot use the processor the other way, i.e., from processed (OUT) to raw (IN). In this case the function "from IN to OUT" is clear but the inverse function "from OUT to IN" does not exist.

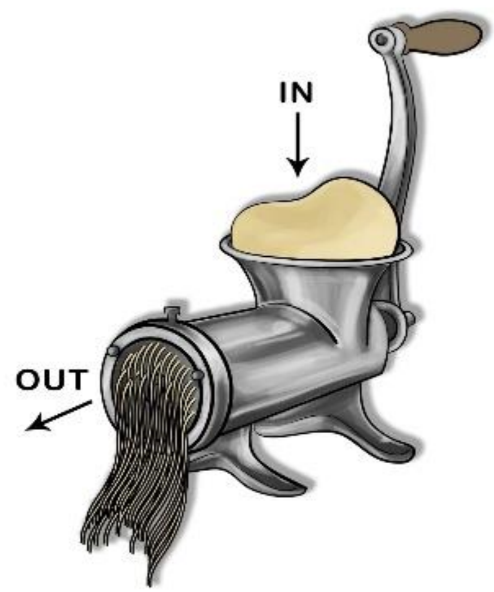

Figure 1: Mechanical meat processor

2.2 Visualizing Inverse function: laptop typing example

Figure 2 reminds us of a daily routine operation, i.e., "cut and paste" or "do and undo" when typing text on a computer screen. This is a visual example that can be used to explain the basic idea behind a function and its inverse.

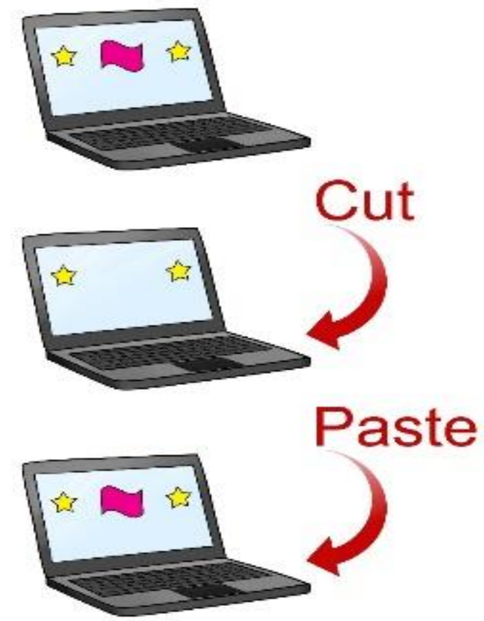

Figure 2: Cut-and-paste example

2.3 Visualizing inverse function: wet paint example

The following sequence of images (Figure 3) can make the concept of function and its inverse very clear to students. Although there are many examples that can explain this concept (such as "negative" and "positive" images in analog "old fashioned" photographic film processing), this one is experience-based with "low-tech" comical flavor: the "function" is an unintentional act, but its inverse is an intentional act with a creative taste.

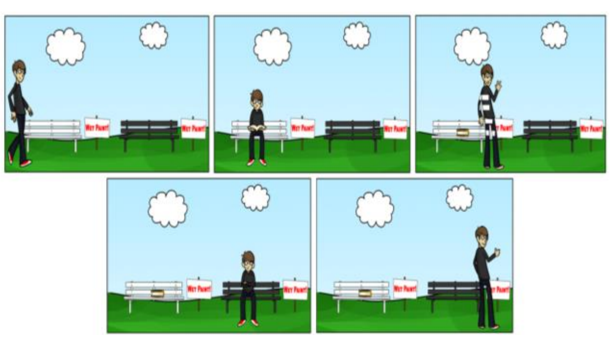

Figure 3: Inversing a "wet paint" effect

The example is inspired by an early 20th century cartoon by Starke, an England cartoonist [23] (Figure 4).

Figure 4: Wet paint example: an inspiring cartoon

2.4 Visualizing function with no inverse: haircut example

Figure 5 is a self-explained case for a function that has no inverse.

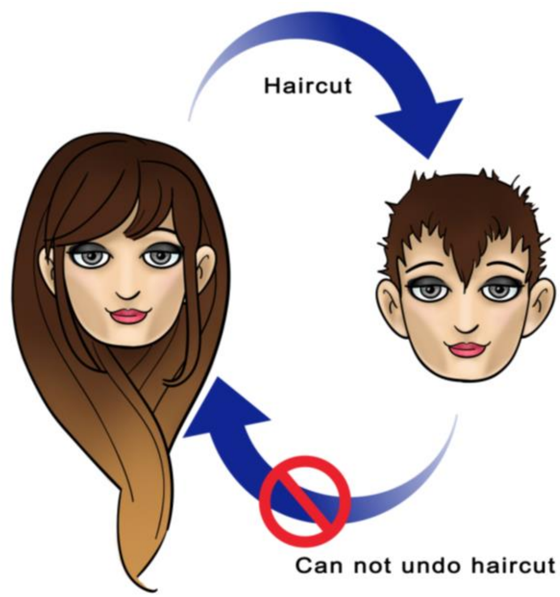

Figure 5: Haircut: the case for a function with no inverse function

2.5 Visualizing "function": mountain trip example

The curve of a mountain's silhouette in Figure 6 can be used to visualize the meaning of "function": for each point on the horizontal axis (x-axis) there is only one value for the corresponding vertical point (y-axis) on the curve. Note that this example can also be used to qualitatively visualize the concept of derivate by referring to instantaneous slopes of the curve.

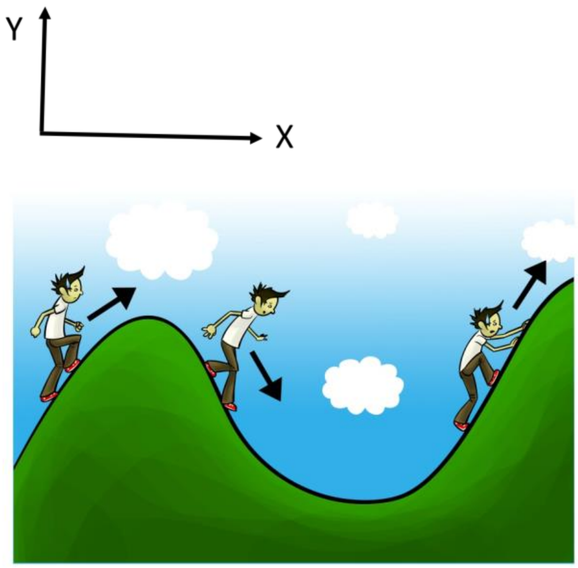

Figure 6: Climbing a mountain

<u>**Venn Diagram Examples**</u>

Venn diagram is a well-known visualization of sets and the relations between them. Venn diagram is very useful when explaining elementary set theory, logic concepts in math (such as *Union* and *Intersection* of sets), relationship in probability, statistics, computer science, and logic gates in engineering (such as *OR*, *AND*, *XOR*). The following example encompasses experience-based relations between shadows that can lead to a better comprehension of the diagram. The example is inspired by the author's work on shadows [24,25] and by observing players' shadows during soccer games.

2.6 Overlapping shadows examples

The following example relates to daily experiences where a Venn diagram can be seen (literally). Observe the following snapshot image taken from a soccer game video (Figure 7). Shadows are casted from multiple light sources leading to multiple overlapping shadows of players. Clearly at overlapping shadow regions, both the first shadow *AND* (*Intersection*) the second shadow exist. Any other region is the *NAND* region, i.e., the complement of the *AND* region. The region where there is a shadow and/or multiple (overlapping) shadows is an *OR* (*Union*) region.

Figure 7: Overlapping shadows

Image source: http://susanreep.com/blog/wp-content/uploads/2010/06/IMG_3497.jpg

Figure 8 shows shadows cast by a cylindrical container from two different light sources (left two images; one source at a time) and from both sources simultaneously (right image).

Figure 8: Multiple shadows of an object

If we take into account the invisible shadows (i.e., the shadows under the container), we can see the following shadow sets (Figure 9).

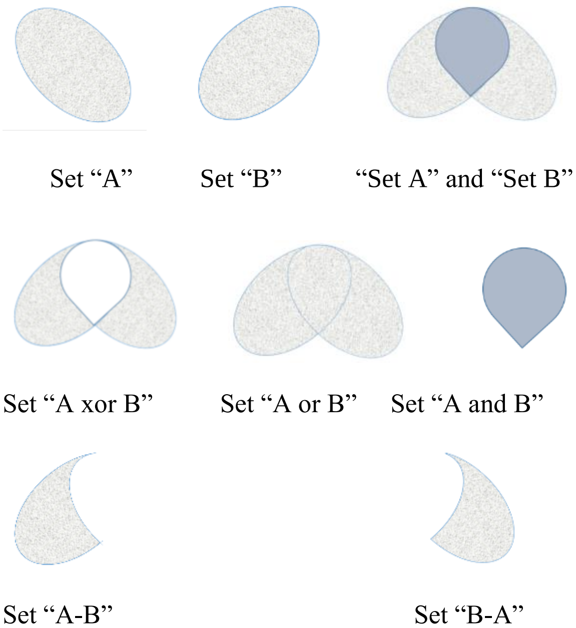

Figure 9: Venn diagram interpretation of multiple shadows. Dark region means overlapping sets.

<u>Large and Small Numbers Examples</u>

It is easy to write down the numbers “one billion” (as 1,000,000,000 or 10^9) and “one nano” (.000000001 or 10^{-9}). But do we have a feeling for what they really mean? The following examples are meant to visualize such numbers.

2.7 Visualizing time examples

Following Admiral Grace Hopper (Navy) approach to visualize a nanosecond [26], we try to visualize a millisecond. But first let’s recall Hopper’s approach: she used to give out pieces of wire about a foot long each to illustrate the eventual problem of building high speed computers. She told the audience that “one foot (Figure 10) ‘is’ one nanosecond,” followed by an explanation that <u>light travels approximately one foot in one nanosecond</u>.

Figure 10: visualizing a nanosecond (scale is in inches)

To visualize a millisecond, observe the final moment in a 100m Olympic game race (Figure 11). Assume that an athlete runs 100 meters in 10 seconds, i.e., averaging 10 meters in one second, or <u>10 mm in one millisecond!</u> A video camera with sampling rate of 1000 Hz (i.e., 1000 frames per second) will snap a picture every 1 millisecond which is equivalent to "spatial distance" of 10 millimeter.

Figure 11: <u>Visualizing a millisecond using distance</u>

Image Source: <u>http://thenetworkgarden.blogs.com/.a/6a00d8341c285b53ef01156f663c92970c-popup</u>

Each millisecond is important. For example, on Aug.15, 2016, during the 2016 Summer Olympics in Brazil, Bahamas' Shaunae Miller (the closer athlete in Figure 48) dived over the finish line to win gold ahead of United States' Allyson Felix in the women's 400-meter final. A few milliseconds made the difference! [27].

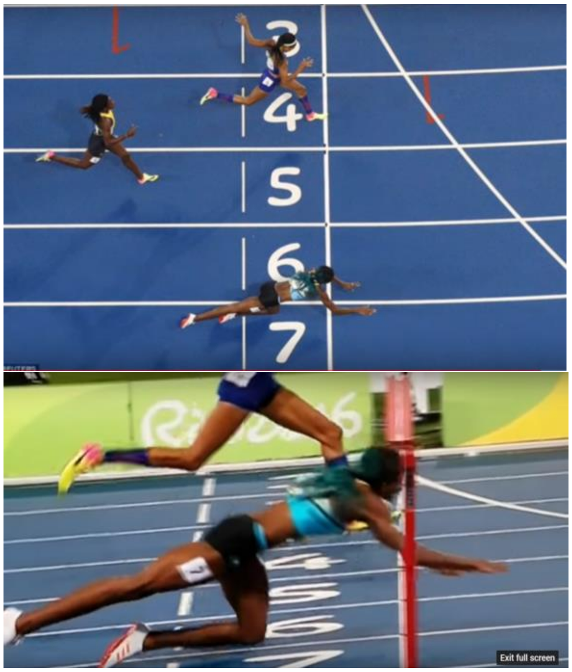

Figure 12: Visualizing time difference
Image source:
https://qz.com/759267/this-is-a-totally-fair-way-to-win-an-olympic-gold-medal-in-track/

It was not the first time a runner dived to win a medal. Look at USA's David Neville (Figure 13) last moment action to win a bronze medal (during the 2008 Olympic Games) [28].

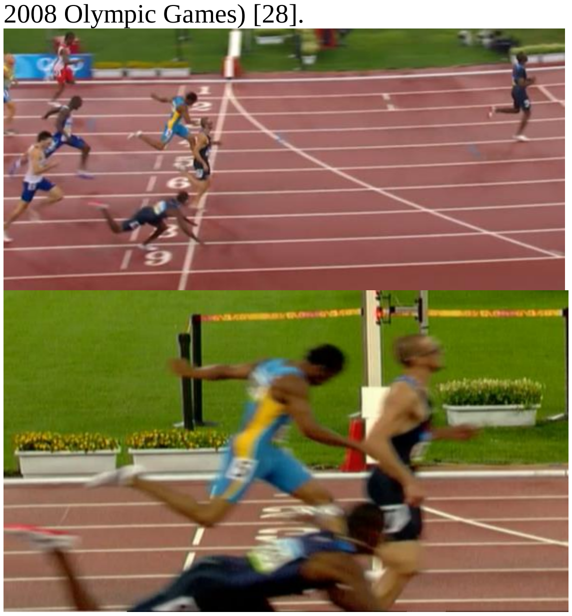

Figure 13: Visualizing time difference
Image Source:
http://www.nbcolympics.com/video/beijing-2008-david-neville-dives-400m-bronze

Figure 14 is a TV snapshot from the 2018 Winter Olympic Games in South Korea (Men's 1500m).
It shows both the time difference (2 milliseconds) and the corresponding spatial difference between the blades.

Figure 14: Visualizing 2 milliseconds

To visualize a microsecond (0.000001 second):
Think of an airplane flying at speed of 250m/sec (multiply be 3.6 to get the speed in km/hour). This means that it moves at 250mm/millisecond or <u>0.25mm (250 micrometers) in one microsecond</u>.

A space shuttle must reach speeds of about 28,000 kilometers per hour (or 7770 m/sec) to remain in orbit. This is the same as 7.77 m per millisecond, or 7.77mm per microsecond.

So space shuttle moves approximately <u>8 millimeters in one microsecond.</u>

2.8 One-billion-dollar example

We hear about large numbers such as one billion dollars ($1,000,000,000, Figure 15), one gigabyte (1,000,000,000) or one terabyte (1,000,000,000,000), but do we really have a feeling for what these numbers actually mean?

Let's focus on one billion dollars. Assume that you work 250 days each year. <u>If you are paid $100,000 per day for 40 working years… in the end you'll have one billion dollars!!!</u> Hopefully it is clearer now.

Figure 15: The meaning of one billion dollars
Image Source:
http://fakemillion.com/wp-content/uploads/2015/11/nm287-Billion-Dollar-Front1.jpg

2.9 LOTTO winning odds example

The numbers of combinations for getting 6 different numbers (aka 6-of-6) out of 53 are [29]:

$$\frac{53!}{6!\,47!} = 22{,}957{,}480$$

This means that the odds of correctly choosing the winning combination of 6 numbers are 1 in 22,957,480. To visualize the odds, multiple analogies have been suggested, for example [30]: (a) getting hit by lightning is almost 4 times more likely than winning the lottery, (b) winning an Olympic gold medal is 75 times more likely than successfully guessing all six numbers correctly, and (c) giving birth to identical quadruplets is also more likely than winning the lottery.

These analogies send a clear message that the odds are very slim, but they are not tangible enough. A more concrete visualization might be:

Imagine yourself in a room (#1, Figure 16) with additional 100 people. In a random drawing you are being picked up as THE winner. You move to another room (#2) where other 100 people have been waiting for you. In a drawing, now taking place in room #2, you are being picked up randomly as THE winner (again). You move to room #3 where other 100 people have been waiting for you. Believe it or not you are being picked up randomly again as THE winner in room #3. Now you move to room #4 with 21 people. A random drawing picks you up again. Very lucky, right?

You have been very lucky 4 times! The odds for being picked up randomly as described above are 1:101x101x101x22 = **1: 22,666,622**

What this has to do with winning the Lottery?

The odds of winning the Lottery are <u>less</u> than the odds of being lucky 4 times in a row as just described.

I wish you lots of luck!

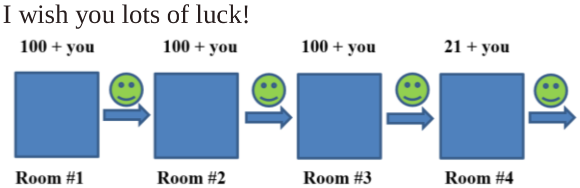

Figure 16: Visualizing odds to win the lottery

CONCLUSION

The illustrated sets of examples attempt to introduce basic math concepts by linking them to daily experiences using relevant analogy-based examples. The idea is to introduce math-less visual and intuitive examples so that students understand and comprehend basic concepts and their relevance. It is important to emphasize that the material presented in this paper is meant to be <u>add-ons</u> to existing calculus textbooks, and is <u>not</u> meant to suggest competition, modifications or replacement of existing textbooks. The presented material is referred to as <u>work in progress</u> and can be shared and discussed with multiple audiences.

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